Władysław Milo

The Fundamentals of Economic Research

The Fundamentals of Economic Research

WYDAWNICTWO
UNIWERSYTETU
ŁÓDZKIEGO

Władysław Milo

The Fundamentals of Economic Research

WYDAWNICTWO
UNIWERSYTETU
ŁÓDZKIEGO
Łódź 2018

Władysław Milo – University of Łódź, Faculty of Economics and Sociology
Institute of Econometrics, 90-214 Łódź, 41 Rewolucji 1905 St.

REVIEWER
Marek Gruszczyński

INITIATING EDITOR
Monika Borowczyk

EDITOR
David Ramsey

COMPUTER COMPOSITION
Maciej Malaczewski

TECHNICAL EDITOR
Leonora Wojciechowska

COVER DESIGN
Katarzyna Turkowska
Cover Image: © Depositphotos.com/peshkova

Printed directly from camera-ready materials provided to the Łódź University Press

Published by Łódź University Press
First edition. W.08460.17.0.M

Printing sheets 19.875

ISBN 978-83-8142-149-2
e-ISBN 978-83-8142-150-8

https://doi.org/10.18778/8142-149-2

Łódź University Press
90-131 Łódź, 8 Lindleya St.
www.wydawnictwo.uni.lodz.pl
e-mail: ksiegarnia@uni.lodz.pl
phone. (42) 665 58 63

Contents

Preface

Being a researcher in the field of economics requires broad cognition of the history of economics, including understanding the development of particular economic institutions and knowledge of historical facts and phenomena. In addition, especially recently, it requires advanced knowledge in the principles of measurement, the gathering, processing and analysis of electronic data, transparent description of facts and phenomena, explanation of data generating processes (DGP), methods of collating facts and making predictions, as well as methods of assessing the quality of the decisions of economic agents.

In order to satisfy these requirements, an empirical economist should aim at being able to build, by himself, empirical forecasting and/or practical decision making tools that are useful for large companies, banks, central banks or government, either at local or central level.

I share the view that in order to formulate a good economic theory or models that describe a given real economy well, an economist, or econometrician, should have long experience in economic modelling and be open to new ideas of economic theorizing and modelling. The advantages of practical experience in dealing with accounting principles are seen in the treatment of equilibria, e.g. in the texts of L. Walras and J. M. Keynes.

Having made the above remarks, it should also be added, that it is important to have access to current knowledge in the fields of logic, mathematics, computer science and statistics. Such knowledge and its repeated use help to develop habits with the goal of writing texts concerning economics at an academic standard close to the standards of the so called natural sciences (e.g. physics, chemistry, engineering).

The use of logic in economics can be easily seen in the texts, of e.g. J. J. Mill, W. Jevons, J. M. Keynes and B. Stigum. The language and rules of mathematical reasoning are commonly used by mathematically or quantitatively oriented economists, such as A. Cournot, M. Allais, L. Walras, A. Marshall, I. Fisher, P. Samuelson, R. Solow, R. Frisch, O. Lange,

M. Kalecki, D. Kreps, W. Eichorn, R. Allen, K. Arrow, G. Debreu, J. Nash, B. Stigum and K. Wicksell.

The practical use of econometrics and statistics is easily seen in the publications of, among others, J. Tinbergen, J. Cramer, O. Krelle, L. Klein, D. Hendry, W. Welfe, A. Baron, W. Maciejewski, W. Charemza, V. Dlouhy, R. Fair, J. Gruber, B. Hickman, J. Kornai, M. Nerlove, Z. Pawłowski, R. Wuandt, I. Sujan, K. Wallis, W. Milo, E. Pentecost and C. Papazoglou.

There is also a group of economists and econometricians that are theoreticians, i.e. they have solid foundations of knowledge in logic, mathematics, economics, and possibly in sociology and social psychology. Having this knowledge and great intuition, they can propose innovative theories and formal models of these theories and sometimes draw very unusual conclusions that are practically verifiable and important. Their propositions are often very influential, both theoretically and practically.

From the point of view of theorizing, modelling, forecasting and decision making, it is very important to recognize and understand the essence of fundamental concepts used in economics.

This monography presents and discusses the following categories of concepts:

- truth and causes (Chapter 1),
- randomness (Chapter 2),
- isomorphism, homomorphism, models, modelling (Chapter 3),
- models, modelling (Chapter 4),
- expectations (Chapter 5, 6, 7),
- instruments of monetary policy, the effects of their use (Chapter 8),
- potentia, calculus of potentias (Chapter 9),
- theory, theorizing (Chapter 10).

The presentation and discussion of these concepts include their historical roots, differences in their interpretation at various periods of time, their role in the development of science, and economics in particular, their usefulness in solving theoretical and practical problems in research and everyday life, perspectives for concrete applications, as well as technical thresholds and barriers to their theoretical and practical use. In addition, the idea of types of knowledge is discussed.

The first three chapters concern the general cognitive foundation of economic research, which is important for further reading and reflection. Chapters 4 and 5 should also be useful for researchers not involved in the field of economics. Chapters 6–9 strictly relate to economics, both in form and content. They present the author's thoughts on the essential purpose of economics and the usefulness of the categories discussed earlier.

The structure of the book is based on the relationship between concepts rather than on particular fields, although this is often not editorially pointed out. The main subjects of the text are the principles of modelling and theorizing and their usefulness to economists, treated broadly from the perspective of the methodology of science. Due to space constraints, some highly important scientific ideas are not addressed here, such as equilibrium, force (also as an economic concept), methods of measurement, mereological principles, games and entropy. However, they obviously deserve deep discussion within the framework of economics.

In the course of writing this book, I was lucky to have the technical support of Anna Krysińska, Dr. Paulina Malaczewska, Dr. Maciej Malaczewski, Dr. Dominika Machowska, Dr. Anna Michalak and Dr. Mariusz Górajski (in preparing the electronic version of the text), as well as Dr. Magdalena Ulrichs for carrying out econometric computations that illustrate the usefulness of statistical measures of potential GDP. I am very grateful to all of them. I express very special thanks to Prof. David Ramsey for reviewing and editorial contribution, which helped to improve essentially the clarity of the text. Special mention deserves Prof. Marek Gruszczyński, the University of Łódź Press reviewer, to whom I am obliged for many suggestions and critical remarks that helped to improve the reception of this book.

Chapter 1

Truth and causes

Summary

This chapter contains a synthetic treatment of the historical evolution of the concepts of truth and cause, as well as some recent conclusions about semantic content in the notions of truthness and causality. It should be stressed that these concepts were used by French and UK economists from the XVII c. and later by, among others, German, USA, Austrian, Spanish, Latin, Polish, Hungarian, Swedish and Italian economists. They used these notions according to their common intuitive meaning. In the English language literature, the first examples of a more formal use of these concepts was probably initiated in econometrics by H. Wold (1960), R. L. Bassman (1965, 1988) and C. W. J. Granger (1969, 1980). Later, the topics of causality were broadly discussed by, among others, K. D. Hoover (1990, 2001), S. F. LeRoy (1995), D. Hendry (1995), C. Sims (1972), A. A. Zellner (1979), J. Woodward (1995). The text presented below refers to some of the author's own texts written in Polish, e.g. W. Milo (2013, 2014), as well as to a very rich literature, mainly in the English and Polish languages. The results of our discussions will be illustrated by considering chosen texts of A. Smith, and remarks are made on the empirical verification of causality and truthness.

Keywords: truth, cause, truthness, causality, criteria for truthness, criteria for causality, causality and correlation.

1.1. Introduction

Truth and cause were and remain categories of notions. They have been the subject of hot debates from antiquity to the present. Such discussions have been carried out by representatives of all fields of science, arts, professions, age groups, nations and races, both theoreticians and practitioners. In economics, the positive role of these categories was particularly underlined by T. Koopmans and H. Simon from the Cowles Comission, as well as, R. Frisch, J. Tinbergen, O. Lange, M. Kalecki, J. M. Keynes, M. Allais, C. Granger, H. Wold, R. Bassman, R. Strotz, and later, e.g. B. Stigum and K. Hoover. Our presentation of 38 notions of truth and more than 30 notions of cause is mainly based on Corpus Aristotelicum written by Andronikos from Rhodos (I c. B.C.), and known to us e.g. from its Polish translation in 9 Volumes with ancient and modern commentaries taken from the appropriate translations into the main European languages. If there are errors in these interpretations, the author takes responsibility for them. It is hard to say today whether our modern understanding of the 7 ancient Greek words for "truth" and 67 words for "untruth", as writes J. Woleński, is exactly the same understanding as ancient Greeks used in day to day life. Similar remarks may be formulated for the words "cause" and "uncause". A growing tendency can be observed, beginning in the XIX c., among both scientists and non-scientists to accept the view that we cannot recognize the precise causes of real life facts, events, phenomena and processes and the truth about them is hidden, latent. Thus the only thing we can do is to estimate the probabilities of their occurrence. This stance has been accepted even by physicists, chemists, engineers, physicians and economists.

Nowadays, in all types of sciences, as well as the life of societies, the language of stochastics is in common use as a way to speak about our ignorance regarding why something happens. The question of how has become more often asked than question of why, because of the ideological impact of the Enlightenment and Positivism, together with the rapid progress in inventing new tools of measurement, production and experimentation. The XX c. technological revolution, resulting in the massive use of personal computers and smartphones, as well as easy access to user-friendly programming packages carrying out logical and mathematical computations and presenting multidimensional graphics, has created natural grounds for making empirical research accessible and enjoyable – research connected with simple questions of how empirical processes behave and what are the chances of their patterns of evolution repeating in the future.

Throughout the history of modern economics, XVIII–XXI c., the question of why has always had an important status. Economists and financists are keen to infer the causes of economic facts, events, phenomena and processes. In this respect, they have used deterministic analysis and, also very often since the 80-ties of the XX c., empirical analyses based on the results of econometric modelling.

Section 2 presents chosen formulations of the notion of truth from the classical and modern eras. In Section 3, we consider formulations of the notion of cause made by the philosphers of ancient Greece and by more modern thinkers. Section 4 considers practical problems involving the inference of the essential causes of economic phenomena using models based on economic theory.

It is hoped that this brief presentation of the unusually rich history of the development of the notions of truth and cause will be useful to economists who have the need to expand their theories or models by including an analysis involving these concepts.

1.2. Notions of truth

It is known that the roots of notions of truth can easily be found in the everyday use of national (or, in earlier times, tribal) languages. Greek uses the word "$\alpha\lambda\eta\upsilon\varepsilon\iota\alpha$", in Hebrew we have "met" and "munah", in Latin it is "verum" or "verite", in German "Wahrheit" and in Polish "prawda". In all of these languages, there are nouns and adjectives that commonly enrich the meaning of the word "truth". In common thinking, the notion of truth still exists as a reference point, despite the fact that post – modernists treat the word in a hostile way.

In order to discuss the concept of truth, it would be instructive to first answer where we should search for and find the truth. There are plenty of fields, places and objects in which it can be found. These are, as follows:

1. The realms of Nature, Science, Arts, Culture, Civilization, Worlds, Universe;

2. Pronouncements, propositions, judgements, sentences, lemmas, theorems;

3. Theories and models of objects, phenomena, processes;

4. Perception, recognition, observation, measurements, awareness, seeing, consciousness, discoveries;

5. Beliefs, opinions, convictions, views, expectations, habits, stereotypes;
6. Feelings, emotions, sensations, impressions, imagination, thrills, will, wants;
7. Descriptions, explanations, analyses, predictions, discussions, corroborations, justifications, proofs, arguments, evidence;
8. Facts, events, phenomena, processes, states of the World, Universe;
9. Acts, actions of the senses, Reason, Mind, human and animal brains, as well as the actions of Nature's elements and particles;
10. Languages (natural, scientific, artificial);
11. Properties, features, attributes of things, objects, processes;
12. Causes, effects, reasons and consequences, relations, functions, mappings, correspondences, homomorphisms, isomorphisms, $A \Rightarrow B$ implications, premises, conclusions;
13. Aims of thinking, acting, and their results;
14. Cognition, knowledge, requirements for: cognition, knowledge, reasoning, analysis, experiments, modelling, theorizing, effective actions, existence of life;
15. Consistency and concordance of thought with reality;
16. Logical value of judgements, sentences;
17. Obviousness, certainty, utility of ideas, notions, principles of thinking, essence of being;
18. Factual understanding of things by observers.

The most general and all-entailing is category (1). It contains both physical objects and their names, as well as, virtual-fictional artifacts.

The most highly psychological are categories (6), (4), (5), which provide both positive and negative stimuli for all the possible activities of economic, political-social agents, as well as scientists.

The categories of notions labelled [2, 3, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18] are especially important for scientists. According to category (9), the truth should be sought by all men, in particular by scientists, artists, and entrepreneurs.

In a certain sense, truth bearers, givers and shelterers or, on the other hand, truth veilers, destroyers, spoilers, humbuggers and misrepresenters are objects belonging to the above categories.

1.2.1. Truth definitions

The following definitions of truth, DT1 – DT26, were adapted from the literature cited at the end of this chapter:

DT1. TRUTH is the well justified, necessary and desired final result of a process of observation.

DT2. SCIENTIFIC TRUTH is a set of reported, observational, factual, indicative sentences and statements about reality concerning nature, society, economics, politics, culture, science, the world, universe, abstract world of ideas and any conclusions logically drawn from them in the form of theses, theorems, axioms, theories and models concerning the quantitative/qualitative relationships between facts, events, phenomena and processes that take place either in the real world or in the abstract world of notions.

DT3. LOGICAL TRUTH is the logical value of propositions, sentences, statements, axioms, lemmas, theorems, scientific laws, theories, models or sentences resulting from the laws of logic based on substituting sentences by expressions of natural or scientific language or truthful logical implications of the type $\alpha \Rightarrow \beta$, α – premiss, β – conclusion.

DT4. ETYMOLOGICAL TRUTH is the meaning, appropriate to the original historical language, of whole pronouncements, propositions, sentences and statements, or parts of theorems and theories.

DT5. RELATIONSHIP TRUTH is the relation between X and Y, where $X \equiv$ thing(s), fact(s), event(s), phenomenon(s) or process(es), and $Y \equiv$ pronouncement(s), proposition(s), sentence(s), axiom(s), theorem(s), law(s), theory(ies) or model(s). A relation can be one of the following: a correspondence, function, mapping, operation, homomorphism or isomorphim.

DT6. LIFE or PRAGMATIC TRUTH is (are) pronouncement(s), proposition(s) or opinion(s) in the form of "life wisdom" that is (are) accepted by the majority of people in a place, region, country or group of countries. These forms of truth are commonly used as the bases for decisions, behaviour, imagination, beliefs, expectations, customs and lifestyle, as well as "unwritten contract(s)" or recorded contracts and accords.

Note 1.1. DT6 should be prominent in empirical quantitative economics, econometrics, social psychology, behavioural psychology and sociology. Some examples of truths of this type are as follows:

- “There are situations when the truth should not be uncovered”, Pittakos (640–569 b.c.). Politicians willingly use this advice;
- “The truth lies in the middle”, Solon (640–559 b.c.);
- “The language of truth is simple”, Eurypides (480–406 b.c.), Aischilos (535–455 b.c.);
- “Useful truth derived from the senses is preferable”, Protagoras (480–410 b.c.);
- “Truthful knowledge comes from Reason, untruthful knowledge from the senses”, Democritus (470–361 b.c.);
- “You cannot recognize truth, while not recognizing cause”, Aristotle (384–322 b.c.);
- “Most people judge others emotionally through feelings of love, expectations, hatred or fear. Only a few judge according to truth or rightness”, “To live is to think”, “The force of habits is great”, Cicero (106–43 b.c.);
- “A greedy man always wants more”, “The masks of truth are misleading”, Horatius (65–8 b.c.);
- “Verum – index sui et falsi”, B. Spinoza (1632–1677);
- “Undiscovered evil grows”, “Work wins everything”, Vergilius (70–19 b.c.);
- “Competition awakens the feeling of indigence and deficiency amongst losers”, Wang Chong (27–97 c.);
- “If language is incompatible with the truth about things, matters do not lead to success”, Kung-fu-cy (551–479 b.c.);
- “Famousness annihilates reverence”, Zoroaster (VII/VI b.c.);
- “The man who recognizes the truth may write it on the table of his heart”, Hebrew Bible.

DT7. RATIONAL ACTIONS, COGNIZANT WITH THE TRUTH OF PRINCIPLES as the effect of discovering the laws governing the existence of things, where principles are abstracted forms of these laws.

DT8. TRUTH IS A REGULATOR, STEERER OF REASONING, INFERENCE, TECHNICAL ACTIONS due to Reason’s need to recognize the priority of prima principia laws which decide about the existence, shape, position and ordering of things in Nature.

DT9. THE CONSISTENCY OF TRUTH is the consistency of the objective state of things in Nature with the inference of cognizant beings in the form of unspoken or spoken thoughts, convictions, dianoic judgements, theorems and theories.

DT10. CAUSAL TRUTH is the corroboration of a causal relationship (logical implication) between real facts, events, phenomena, processes of Nature or between abstract notions of them.

DT11. TRUTH $\equiv$ $\pi\rho\acute{\alpha}\gamma\mu\alpha$ $\equiv$ $\alpha\lambda\eta\upsilon\varepsilon\iota\alpha$ $\equiv$ τo $\ddot{o}\nu$ $\equiv$ τo $\ddot{o}\varsigma$ is a thing, being, fact, event, system of events, deed or action which can be denied only in thought.

DT12. TRUTH is a property of non-contradictory thinking and judgement by an individual about oneself at a specific time.

DT13. TRUTH is that which seems clear to everybody according to their senses (Protagoras).

DT14. FACTUAL TRUTH is the factual understanding of things (Anaxagoras).

DT15. CORROBORATIVE TRUTH is the result of verifying the properties of things, which contains the minimal number of incorrect perceptions about the facts connected with these things, despite possibly describing the greatest number of properties attached to these things.

DT16. TRUTH is the effect of correct reasoning according to logical rules connecting thoughts about an object with the object itself or relating a subject to predicates about that subject.

DT17. TRUTH is the purpose of thinking, acting and selecting the means necessary to realize a given aim.

DT18. TRUTH is the phenomenal nature of a thing determined by its shape, size, mass, structure, colour, motion or degree of content.

DT19. TRUTH is the effect of using correct perceptions and premisses based on empirical knowledge to infer causes.

DT20. TRUTH is the composition of persistent quantitative and qualitative properties of things, natural phenomena, relationships between things, phenomena, or the properties of notions and their essential dependencies, including their compatibility.

DT21. TRUTH is the obviousness of the existence of physically nonsubstantive ideas regarding things, objects of mathematical and logical knowledge, as well as causes, to thoughtful beings.

DT22. TRUTH is certainty, persistency, rightfulness of judgments, rules of reasoning.

DT23. TRUTH is the durability, constancy of an object's existence.

DT24. TRUTH is factuality, testability, certainty, credibility, obviousness, simplicity, exactness of knowledge.

DT25. TRUTH is what an honest, sincere, noble, faithful man says or does.

DT26. TRUTH is the revelation of God and his prophets.

Definitions DT1–DT24 have, among other things, the following features (where V – G denotes Verum – Givers, and V – B denotes Verum – Bearers, and the symbol V denotes truth):

- They are adaptations of texts from ancient philosophical and religious literature (e.g. Corpus Aristotelicum, Hebrew Bible), as well as modifications of these ideas proposed by thinkers and scientists in later periods.
- Thanks to growing awareness of the great ancient civilizations of India and China among their modern representatives, it can be seen that similar thoughts originated there with a slightly different distribution of accents.
- In the class V – G we can distinguish the following "truth givers": God(s); masters; teachers; gurus; prophets; shamans; philosophers; scientists; observers. The definitions DT24, DT25, DT26 are directly or indirectly connected with them.
- In the class V – B we can make distinctions between and connections with appropriate sets of definitions: properties of Nature (DT1, DT11, DT23, DT25), properties of thoughts (DT1, DT11, DT23), results of perceptions, observations, testing, corroborations (DT1, DT2, DT10, DT13, DT15, DT19), natural and scientific languages, theories, models (DT3, DT24), relations of things and processes to sentences (DT5, DT9, DT12, DT22), common beliefs, opinions (DT4, DT6, DT9), laws and principles of thinking, acting and reasoning (DT7, DT8, DT10, DT12, DT14, DT16, DT21, DT22), aims of thinking and acting (DT17), the essence of the properties of Nature, and essence of notions (DT10, DT15, DT18, DT19, DT20, DT21, DT22, DT23, DT24).
- Economists should be interested, possibly to varying degrees, in all of the definitions from class V – B.

It is worthwhile to underline that the above definitions of truth do not exhaust the list of existing or pertinent ones. We may describe, e.g. the following various types of truth:

– necessary, sufficient V,

– hidden, unhidden V (covered, uncovered, transparent V),

– conditional, unconditional V; discovered, undiscovered V,

– doxastic, epistemic V; understandable, non – understandable V,

– trivial, sceptical, untrivial, dogmatic V,

– inductive, deductive, reductive, abductive V,

– valid, non – valid V; empirical, theoretical V,

– contextual, non – contextual V,

– exact, approximate V; actual, realistic, non – actual V,

– general, particular V; relative, non – relative V,

– semantical, syntactical V; factual, artefactual V,

– useful, instructive, currently unuseful V,

– current, eternal, old, novel V,

– time, space V; Reason, Mind, Senses V,

– perceptions, observations, description, explanation, analysis V,

– contingent – positive, contingent – negative V.

There are many notional and practical connections between these definitions and later chapters of this book. In particular, many of the ideas in Chapters 2–7 relate to definitions DT1–DT3 and DT7–DT24.

From the Middle Ages, great, albeit somewhat brief, popularity was gained by the almost propagational statement “VERITAS EST ADAEQUATIO REI ET INTELLECTUS” proposed by the neo – platonic author I. Israeli in the VI c., i.e. TRUTH is what Intellect poses on equal terms with being. As an individual, one can make a pronouncement, proposition, sentence about being, a fact, event, phenomenon, process or material object. Thus his Intellect is a truth maker, and an object is a truth bearer. Gramatically, truth is a noun (substantive). In everyday life, science and the arts, it is also very often used as an adjective as a form of saying that something is truthful. This indicates that the thing considered is certain, obvious, confirmed, corroborated and/or acknowledged by experts, authorities in the field to which the thing belongs. In general, this thing is the object of perception, observation, description, explanation, analysis, prediction, as well as decision making and realization. Another form of expressing thoughts connected with truth is the substantive (noun) form “truthfulness”. This form expresses a property or properties of the thing or things being analyzed, i.e. attribute(s) of object(s) of perception(s), description(s), explanation(s), prediction(s) or decision(s). A question arises as to whether a property or properties should be treated as being currently observed by a concretely existing person for a concretely existing object of Nature in a fixed place and moment of time or as a property per se manifesting the essence of the thing being discussed. It is used, therefore, with regard to the truthfulness of pronouncement(s), proposition(s), statement(s), sentence(s), lemma(s), theorem(s), model(s) and object(s) of Nature, while being formulated by a given person(s) at a given time and in a given situation.

The notion of truthfulness (Verum) was used very often in the Hebrew Bible, Talmud, philosophical discussions by philosophers in ancient Greece, and later, by among others, French, German and UK-philosophers, such as Descartes, Leibniz, Hume, I. Kant, B. Bolzano, G. Frege, B. Russell, and authors later developing these ideas in the XX c.

For example, Descartes distinguishes between, on one hand, the meaning of open and clear truthfulness compared to covered and unclear truthfulness, and, on the other hand, the truthfulness of things cognized by senses or by Reason. For him, the truthfulness of a proposition or statement lies in their certainty, clearness, explicitness and distinctness. He intrinsically assigns to ideas, understood as thoughts or images of things, the feature of truthfulness. But in other cases, ideas can be truthful or false. He assumes, that the will and feelings are always true.

For Descartes the criteria of truthfulness are e.g.:

- maximal clarity of perception (by the Intellect), cognition, understanding, explanation, proof;

- maximal probability of the conjecture’s truth;

- maximal belief compared to alternatives, minimal doubts compared to alternatives;

- maximal obviousness of the considered thing, topic or chosen rule, the possibility of checking Verum;

- minimal error of closeness to the truth for the chosen rule of verification.

He did not make any comparisons between these criteria within the distinguished kinds of truthfulness of, e.g. perception, cognition, recognition, understanding, thinking, description, explanation, proof, conjecturing, analyzing, predicting.

There are some indirect links to some of his opinions on the quality of rational reasoning to assert the truthfulness of physical objects, as well as geometric and algebraic objects, i.e. abstract objects of mathematical thought.

Some interesting comments on truth and truthfulness may also be found in his book "Tractatus de Intellectus Eniemdatione". He discusses the truthfulness of knowledge, good, evil, happiness, things, ideas and the essence of human nature. Truthfulness is, for him, the effect of cognizance of knowledge, non-constant according to temporal and random experience, the content of notions of res, reasoning either from the most direct cause to effect or from general to particular, proportionality-driven structural features, as well as correct deduction, differentiation, consistency and the use of the best possible methods to corroborate conjectures.

Baruch's criteria for truthfulness are not so explicit as Descartes' rules. They relate mainly to objects (i.e. concepts of things, their forms and the laws governing them, acts, facts, notions, closest or non-external accidental causes), or methods for confirming the truthfulness of ideas concerning the things that are being investigated. Among the conceptually expressed criteria for the truthfulness of ideas, he proposed the following: an idea's simplicity, clarity, non-fuzziness, cause-effect relation, onticality, particularity of concepts, understandability of sentences or events, i.e. the clearness of words and definitions of objects. Based on these criteria, he proposed that in practice a cause-effect type of reasoning should be preferred and, when necessary, clearly connected abstract notions should be applied.

It is important to underline the possible direct or indirect influence of B. Bolzano's "Grundlegung de Logik" on G. Frege, B. Russell, J. Łukasiewicz, K. Twardowski, R. Suszko, A. Tarski and scientists in the Vienna Circle. Particular fields of science are treated by Bolzano as collections of truths that have already been uncovered or are aimed to be uncovered. Logical relations and the attributes of truths are sought in sentences per se and their components, i.e. ideas per se. For the latter elements, he distinguished such relations as compatibility (incompatibility), inclusion (all objects of a given idea belong to another specified idea), equivalence (two-way inclusion), subsumption (one-way inclusion), contradiction, opposition (non-compatibility that is not a contradiction), overlapping (compatibility without inclusion). His concept of validating the power of sentence (universally or partially) was especially innovative, as were his definitions of truthfulness relations between sentences that cover the semantic content of words. Bolzano describes text which is "Gültigkeit" or "Allgemainegültigkeit", meaning generally, logically acceptable in a modern sense. According to Bolzano, truthfulness has the following meanings in common usage:

- the "attribute of a sentence(s)" thanks to which a sentence expresses something as it is; this meaning is abstract and objective in Bolzano's understanding;
- concrete and objective existence of the "sentence itself";
- subjective "correct preposition" containing truth in a concrete, objective sense;
- as a representation of a collection of many truths in a concrete objective or subjective sense;
- realness, authenticity of the considered objects, of judgement, perception and explanation (not an appropriate use according to Bolzano).

He distinguishes the concept of "truth in itself" as being different from the concepts of "cognized truth", "certainty", "reality", "real objects", "cognizability" or "possibility of thinking out". When he critically discusses different concepts of truth, he sees the positive features of the concept of formal truthfulness, understood as sentences being non-contradictory, together with their inferability and compatibility. Thus rules that satisfy the cardinal Bolzanian requirements may be treated as criteria for checking truthfulness in the Bolzonian sense. These criteria may be treated as precursors of, e.g., G. Frege's and A. Tarski's criteria.

As a mathematician, logician and philosopher of language, G. Frege was influenced by G. Leibniz, I. Kant, F. Gauss, F. A. Trendelenburg and, in particular, R. Lotze (see, e.g. H. Sluga, D. Bell, A. Gut). According to Sluga, Frege uses Lotze's ideas of objectivity, which state that propositions (Urteil) are more important than concepts. Hence, the principle of context belongs more to the theory of propositions than the theory of meaning. This promotes the role of a priori cognition due to the limitations of empiricism, the potential of applying the mathematical concept of function and its arguments to logical interpretations using suitable symbols (note: G. Boole, de Morgan introduced algebraic conventions and rules to logic, while Lotze and Frege introduced the main object of mathematical analysis, i.e. functions), and the idea that mathematical concepts are close in form and content to logical concepts, so mathematics can be treated as a branch of logic. The last idea resembles the similar ideas of B. Russell and A. Whitehead, which developed later. Frege's very important contribution to truth theory is the statement that the truthfulness of a sentence depends on the semantic properties of the parts of the sentence considered, i.e. the role of the formal proper names used to represent the objects, and the role of the predicate part of a sentence. The composition of these two characteristics of a sentence determines its meaning and context of meaning.

Distinguishing theory of the senses from theory of meaning (reference) and treating the concept of senses as being connected with the logical concept of truth leads Frege to the idea that the laws of logic are semantic laws about truthfulness. Thus, the Fregian criteria for truthfulness may be formulated as logical criteria for truth or truthfulness and checkable semantic certainty.

Bolzano's and Frege's ideas were later developed by members of the Vienna Circle, as well as their correspondents, e.g. A. Tarski and C. Popper.

Tarski's texts are mathematically and logically very precise. His definitions of truth use, either indirectly or directly, a definition of the order (degree) of a language. Apart from common languages, he defines meta languages of integer order (and infinite order). One of his preferred definitions of truth is "p is a true sentence if it is a thesis of science, i.e. it is translatable into the meta language of a given scientific discipline". Another of his definitions states that "p is true if it is formally provable in a scientific meta language and is semantically fulfillable". Tarski's criteria of truthfulness are therefore criteria belonging to formal logic, i.e. they are rules and laws of mathematical logic.

Popper uses Tarski's arguments, while trying to describe a theory of truth in terms of correspondence with facts. This view has been criticized by many logicians and philosophers (for a presentation of suitable discussions see K. Niebrój (2010)). However, Tarski avoids such terms as facts, correspondence and reality when defining truthfulness.

Popper tends to avoid definitions, or the meaning of words, yet he accepts Tarski's definition of truth. However, Tarski does not define the notion of the content of truth, neither truthfulness as a feature of a sentence, but the set of true sentences within a meta-language. Correspondence with facts, non-linguistic facts, and the conditions of this correspondence are crucial in Popper's approach and can be treated as the basis for defining Popperian type criteria for truthfulness.

This means that Popperian criteria are rules for confirming that the description of an investigated object, term or property corresponds to concrete facts according to the available evidence, so that a correspondence relation is in effect. In this respect, Popper takes an opposite view to Frege, who strongly criticized any theory of truth based on correspondence. Taking into account A. Grobler (2006), Nowaczyk (1998, 2004) and Woleński (2005), it should be said that Tarski presented truth in terms of a given language and given interpretation with the assumption that each sentence signifies something, i.e. a sentence has one meaning and that any mapping of words (terms) from a given language into words (terms) of a given meta language preserves their meanings. Tarski's definition of truth does not define the content of this concept, but only an extension of it and, as Szymura (1995) says, this definition is not appropriate with respect to the truthfulness of scientific theories. Though Descartes and Locke enriched the notion of truth with the concept of correspondence, G. Moore, B. Russell and Wittgenstein gave it a stronger coherence. For Russell, the correspondence of propositions to reality consists in the consistency of beliefs with some considered complex of real objects, and for Wittgenstein, correspondence means mapping a proposition's structure with respect to the structure of the object that is of concern in the proposition. In the author's opinion, definitions of truth using this meaning of correspondence should be a subject of deep discussion by economists.

Despite the fact that economics based on factual correspondences would have a structure that is so complex that it excludes the construction of structural models isomorphic to them, the author believes that constructing an approximate counterpart of such ideal models is valuable, both theoretically and practically.

Logical empiricists are convinced that in order to determine the truthfulness of sentences using empirical confirmation, it is not sufficient just to use additional logical criteria. As Neurath and Hempel state, the truthfulness of scientific sentences, due to their generality, means more than the truthfulness function of individual sentences and the coherence of their contents. For example, van Frassen states that there is no need to talk about the truth or truthfulness of a scientific theory. It suffices to check whether it is acceptable (empirically appropriate) on the grounds of chosen criteria for the predictability, explicability, describability of real world phenomena. Davidson and Putnam, in turn, accepted the thesis that theory is not determined by empirical data, so according to them, criteria for the rational acceptability or approximative truth of a theory are sufficient, which leads us to suitable scientific procedures or methods accepted by a large group of scientific experts.

In this respect, Popper's concept of verisimilitude has been also intuitively broadly discussed and used in analyzing the properties of theories or scientific knowledge. Truthfulness is replaced here by either the validness, acceptability, testability, confirmability or justifiability of sentences and theories (see, e.g., Newton, Smith (1981), Putnam (1981), Davidson (1967, 2001), Szubka (2003) and B. Tuchańska (2012)).

According to G. Hegel, K. Marx and F. Nietzche truth emerges as the result of the social-historical context of interactions between truth as a human value and the socially-historically rooted and felt imperatives of human behavior to which the following belong: the need to act, cognition of the world, obtaining power and capital. It follows from such views (see Allen (1994)) that there are no good arguments for accepting a particular truth.

Modern philosophy, being under the strong impact of the theoretical and technical achievements of the XX c., is trying to absorb the criteria for truthfulness applied in the natural sciences (especially physics), mathematics and logic. This means, that the final empirical criteria for confirming the truthfulness of scientific statements are the results of empirical observations, i.e. empirical measurements derived from direct observations or suitably controlled experiments. In the case of economic agents, there is no possibility of satisfying the assumptions of controlled natural experiments, so economists are placed in a situation of having to use so called model-based computer simulations, and hope that the numerical results from these simulations are close to quantitative observations concerning relevant economic variables. Well-known economic theories and their approximations now mostly use a special meta-language, which is a mixture of a common national language, the scientific languages of mathematics, logic and economics, and possibly even languages from such fields as physics, biology, sociology, psychology and history. Due to the uncertainty of measurements, economists are now (since the 80's of the XX c.) commonly using the language of stochastics and probability to express their thoughts about economic phenomena (there is more discussion on this in later chapters).

1.2.2. Determinants of quality of truthfulness criteria

The validity of using particular science related criteria for truthfulness may be discussable due to:

- arbitrary or vague arguments that are aimed to defend the choice of a given criterion;
- using categorical statements that are not always true by definition, based on threshold quantities which help to confirm hypothesis(es);

- non-unique definition of categorical statements regarding the conditions for accepting or rejecting the hypotheses to be tested (logically, epistemicly, empirically, experimentally);
- categorical statements are usually contingent on essential features of the objects studied (e.g. physical objects, pronouncements, propositions, sentences, models, theories, descriptions, explanations, proofs, analyses, predictions, plans), and in some cases they are not truthful sentences (see e.g. the discussions made by J. Koethe, S. Kripke, J. Canfield);
- most economists, sociologists, psychologists use their own mixed-type of language with statement-generating terms, such as "it is necessary that", 'it is sufficient that", "it is possible that" and their negations. Hence, the laws of logic provide conditions for the truthfulness of specified propositions, sentences or theories, where the automatic checking of their truthfulness is understood as computing the validity of a model or correctness of a theory via sequential checking that the assumed conditions are fulfilled (a computational version of the modal laws of logic are discussed, e.g. by D. Chapman, M. Fox and D. Long);
- truthness of knowledge propositions and sentences depends only on the, assumed to be true, context-relative propositions and sentences that fulfill all the necessary and sufficient requirements of a suitable method of corroborating the truthness of the sentences considered; thus truthness is contextual within the framework of existing knowledge and influenced by the selection criteria adopted;
- economists, psychologists, sociologists and historians use, consciously or unconsciously, the ancient mathematical method of proof by contradiction (i.e. assuming that a statement is false and showing that this leads to a contradiction), when they accept quasi-factive approximate reports of empirical measurements which have the form of false propositions. As M. Shaffer (2015) argues, there are specific conditions under which such false propositions can constitute evidence for the truthfulness of other propositions or sentences;
- increased stress has recently been placed, following Chrysippus, on the great importance of the meanings of words belonging to natural languages (or their modified definitions in first or second order scientific languages) (see, e.g., the remarks of B. E. McDonald (2000)); however three criteria for forming a good scientific theory, within the chosen

mixed-type language, are too general to be instructive as to how to do it in practice, despite their very good mathematical and logical foundations;

- the methods of graphical representation and analogy, e.g. the stick-and-ball model of a molecule or graphical-symbolic models of a firm's structure, are imperfect meaning bearers, not only for conceptualizing real objects, but also as grounds for essential discussions regarding important, practically valid, empirical processes. However, due to the complex and necessarily precise actions of gathering, classifying, storing, and analyzing collections of research data, results are not always comprehensive, coherent or reliable;

- the internal logical consistency of a set of sentences (e.g. a theory T) does not imply the truthfulness of T (see J. Woleński, 2010). For economists, this insufficient criterion should be treated as only one of the criteria to be applied with respect to T. Hence, verifying consistency should not simply check internal logical consistency;

- it is difficult to compare possible choices for quantifying truth values from the standpoint of probabilism, credence theory, belief functions and fuzzy set theory, or their impact on the choice of quantified criteria of truthfulness from the point of view of truth makers, truth users, truth bearers, truth evaluators, truth tellers and truth explainers;

- conceptual changes in the meaning of concepts, new articulations of scientific hypotheses (not only in economics), accumulation of new empirical evidence that enable the formulation of new hypotheses about data generating processes representing fictional or real world mechanisms, the existence of many rules of inference and many good indicators of the truthfulness of a research hypothesis may lead to situations called a "horn of plenty" (see e.g. G. Betz (2015), A. Goldman (1999), J. Pollock (2001)), and problems with the logical consequences of practical distinctions between the notion of "evidence for a scientific proposition" and the concept of "evidence for a real state of affairs", where the first is not factive, but the second is factive (see e.g. A. Heathcote);

- it is possible that the definitions of quantitative indicators of the truthfulness of economic models and theories based on the structural characteristics of their validity taken from an unsigned (or signed) semantic decomposition tree (i.e. semantic truth tableaux) and their transformations into a programming package for logical computations would help researchers in the future to check and improve their models and theories in reference to their truthfulness;

- formulations of practical and valid indicators of the truthfulness of the results of research, or of empirical models and theories, depend heavily on scientific understanding that should be factive (e.g. take into account facts regarding a concrete economy), where understanding is assumed to be a skill of a scientist or decision maker, but not his knowledge about the facts derived from a model or a theory assumed to be true or approximetely true; this does not exclude the possibility of a practitioner using fictional models or theories of real phenomena as a (theoretically well founded) reference point when making comparisons between different empirical measurements of real phenomena (see e.g. H. de Regt or Chapters 4–7 of this book);

- it is discussable as to whether the truthfulness of a proposition is independent of the convictions, cognitive habits, ways of understanding the content of the proposition, ambiguity of language or anything existing exogenously with respect to an observer who makes his utterences and expresses his propositions, sentences, theorems, models and theories;

- is it valid, as K. Twardowski says, that relative truths and truthfulnesses have their defenders only thanks to placing equivalence on the contents of the notions of differing propositions and pronouncements, not ascribing them precise, strict content, and not accepting the principles of contradiction and the excluded middle?;

- there are, among others, two important views for economists: one states that what makes an assertion or proposition or sentence truthful is not its properties, but how closely it reflects the real world and the second, that there is no such thing as the truthfulness of propositions or sentences independent of economists' theories, notional systems and language. Criteria for asserting truthfulness based on these views will be different;

- there are no systematic studies of the impact of the epistemic consequences of a determinator's, modifier's, trivializer's use of the adjectives "truthful", "false", and their corresponding nouns for making comparisons between the epistemic qualities of different scientific models and theories;

- a question arises as to whether it may happen in the future that human beings as a whole, and scientists as a particular group, reject the practical use of logic based on the value representing truth as the driving value and replace it by the value representing falsehood (logically and theoretically this is possible), and whether such a change would promote the development or degradation of economics and societies;

The above list of uncertainties regarding the consequences of so much ambiguity in the meaning of the truth and truthfulness is not complete. The following are among the many criteria for assessing the truthfulness of propositions and sentences, which may be especially suitable for economists:

- the ontic correctness and robustness of empirical economic models of concrete economies;
- semantic and syntactic cautiousness and respect for using the proper meaning of economic concepts and concepts from other fields of science;
- clear use of the rules of logic and mathematics in formulating the form of economic models, theories confirmed by competent reviewers;
- constant verification of economic sentences using new economic facts, new factual perceptions about economic events, phenomena, processes and new behavior patterns among economic agents, as well as checking these sentences scientific validity and empirical truthfulness;
- clear and effective use of new methods of perceiving, observation, experimentation, description, explanation, verification, in order to increase the reliability of checking, discovering, confirming the truthfulness of sentences, theories of the functioning of economic units and entities or theories of economic processes.

Summing up our reflections about the conditions and criteria for assessing truthfulness in the context of the 21 definitions of truth given above, it should be said that:

- the use of all the definitions (except perhaps DT13) requires the application of logical criteria for truthfulness;
- the use of definitions DT1, DT2, DT5, DT17, DT20, DT21 requires the application of mathematical and statistical criteria for truthfulness;
- the use of definitions DT4, DT7, DT16, DT18–DT21 requires application of truthfulness criteria from analytical, ontological or traditional epistemological philosophy;

– the use of definitions DT2, DT5, DT6, DT10, DT17–DT20 requires application of truthfulness criteria that are popular within particular fields of sciences.

It is worthwhile to recall a very popular (from the Middle Ages to the present day) brief summary of the acknowledged content of the notion of truth given by I. Israeli (VI c.) in the following form: "Veritas est adaequatio rei et intellectus", i.e. truth is the correspondence between intellect and a thing (a subject of observation, analysis and cognizance).

To close this section, we propose taking into consideration the first two chapters of A. Smith's (1776) original version of "Inquiry into the Nature and Causes of the Wealth of Nations". He used the common meaning (at his time) of many economic, historical, legal, philosophical and ethical concepts, including the notion of truth and cause. His thoughts and propositions, are indirectly connected with DT2, DT5, DT6, DT9, DT13, DT15, DT17 and DT19. Although there are a great number of suitably given economic and historic facts in the cited chapters and others, I must underline that Smith's analysis of the factors of growth retains its validity as an empirical model, despite e.g. the great difficulties in measuring the Polish economy of the 90's of the XX c. (see the results of modeling Polish growth in W. Milo (2002), Chapter 3).

More historical remarks on the evolution of the concepts of truth and truthfulness, as well as their applications, can be found in the literature given at the end of this chapter.

1.3. Notions of cause

The category of "cause" is, in many ways, connected with the category of "truth". The following two sentences of warning are probably the most contentious and meaningful with regard to this relationship. They are:

*1. You cannot cognize truth without cognizing cause;

*2. You will cognize truth if you cognize cause.

The first sentence is well known from the α – minor book of Aristotle. The second, also well known phrase, is a simple transform of (*1). Both categories, depending on one's approach may concern either physical objects, their states, events, processes, phenomena and facts connected with these objects, or abstract objects like pronouncements, propositions, sentences, models, theories, ideas and concepts that can describe real physical objects or fictious, virtual objects.

It should be stressed that the problem of causality was posed in ancient times by many thinkers, especially by Greek philosophers. Unfortunately, apart from the texts of Plato and Aristotle, which were rewritten numerous times, and much later, of Sextus Empiricus, as well as the Hebrew Bible and Talmud, there are no relatively full ancient texts treating the topic of causality more or less broadly. It is easy to find that, both in antiquity and later periods, most discussants openly or indirectly accepted the role and usefulness of the concepts and rules of causality, at least in a moderate form. There were, however, some sceptics (who had their precursors like Protagoras, Gorgias and their followers up until the XX c.) who possessed the view that finding truth and cause is either very difficult or impossible, and some of them propagated the view that these concepts are useless in practice.

An active return to the problem of causality took place in the XVII c., due to the renaissance of the sciences, especially the natural sciences. It suffices to read the writings of R. Descartes, J. Locke, G. Leibniz, I. Newton, B. Spinosa, D. Hume, I. Kant, B. Bolzano, G.Frege, B. Russel, E. Husserl, K. Twardowski, J. Łukasiewicz, A. Tarski, L. Wittgenstein, S. Leśniewski, P. Frank, R. Carnap, C. Hempel, W. Spohn, B. Skyrms, D. Bohm, M. Bunge, P. Suppes, H. Simon, H. Blalock, N. Cartwright, M. Tooley, H. Reichenback, T. Czeżowski, G. von Wright, Z. Zawirski, W. Krajewski, J. Woleński, G. Svechnikoff, and many others not mentioned due to space limitations and language barriers. Some of these authors owe much of their renown to the books (written in German) of W. Windelband, A. Lang, C. Göring, C. Hartenstein and C.V. Peter, which give broad historical descriptions and explanations of the development of the concepts of causality up until the end of the XIX c.

Before we start to discuss the essence of the concepts "cause" and "casuality", it would be instructive to introduce some simple definitions of terms that are often used in all the sciences, terms such as: state of a thing (res), state of a system, event, process, fact, phenomenon, situation.

D1. A state of a thing (object, system) at a given moment, place, situation, is a feature (qualitative or quantitative) attributed (ascribed) to the thing (object, system) being considered, analysed, modeled, discussed by one (or multiple) observer(s), investigator(s), modeler(s), theoretican(s), where this feature is valid at a given moment of time t_a and place $pl(t_a)$.

Remark 1.1. The things (objects, systems) referred to in this and other chapters of the book may signify either physical (non-living or living) objects, or abstract (fictional, virtual, artificial) objects like e.g. mathematical or logical objects. These feature(s) may be perceivable, perceptible and observable (by the five senses of living beings) traits, imaginable by the human mind or result from processing by the human brain of either noticeable external or internal part(s) of physical thing(s) or the appropriate partial or general properties of the semantic content of words, concepts, ideas, propositions, sentences, models, theories, poems, novels or sculptures, which are abstract objects or creations of thought. These features, attributes, properties of things may take qualitative or quantitative forms (according to the tools, abilities and needs of observers, scientists, artists, engineers, physicians, economists, businessmen) and be perspectives of observations, aims at description, explanations, analyses, predictions or decisions. Things and features may be accounted for either in isolation or in relation to other things. Features (properties) may be simple or complex, internal or external, short-term or long-term, partial or holistic.

Remark 1.2. In physics, the macroscopic state of a physical body is described by a set of the values of the chosen parameters of a state which describe that body (e.g. these parameters could be volume, pressure, temperature and internal energy of the body measured at given time t_a, in a given place and situation).

The content of the concept of event may be characterized as follows.

D2.1 An event E_0 is a part of phenomenon, process, phenomena, processes connected with the observed, analysed object(s), thing(s), system(s).

D2.2 An event E_0 is something that is or was correspondent with or concomitant, correlative to definite dates, places, positions (in space or situations)

D2.3 An event E_0 is an element of the set or space of events (elementary, complex, random, non-random).

D2.4 An event E_0 is a state s_j or a set of states $S \equiv \{s_j, j \in J\}$ that qualitatively or quantitatively describe features of observed and analysed object(s) $\{Ob_i\}$, thing(s), system(s) to be characterized at a particular (virtual or actual) moment or period t_a, and at a definite places $pl(t_a)$ or position $pos(t_a)$ in space. It should be noted that in D2.4, f.e.

$$s_{ji} : Ob_i, t_{a,ji}, pl(t_{a,ji}, Ob_{ji}(t_{a,ji})), pos(t_{a,ji}, Ob_{ji}(t_{a,ji})) \to \mathbb{R},$$

where $\mathbb{R}$ is the set of real numbers.

Remark 1.3. Denoting the considered feature (which may be variable or invariant in time) by f, we may also define its values to be determined by the triad (t, $pl(t)$, $Ob_i(t)$, $\{Ob_j(t), j \in J\}$), depending on whether we assume that the impact of neighboring objects is significant. The f depends on observable features, denoted by $\{X_j(t)\}$, and unobservable features, represented by $\Xi(t)$, $t \in T \equiv (t_a - \tau, t_a + \tau)$.

Thus, the definition of the event E_0 requires more formal components strictly related to the state $s_i(t)$ of the object Ob_i, that is to say the following components: spatial location of the objects $\{Ob_j(t), j \in J, t \in T\}$ denoted by $\{pl_j(t)\}$, essential and observable features characterizing $\{Ob_j(t)\}$ denoted by $\{X_j(t)\}$, features that are not included in the current considerations (in a theoretical analysis of a model) represented by the symbols $\Xi_i(t)$, the formal shape f_i of the relation between the variable describing the studied feature $Y_i(t)$ of $Ob_i(t)$ determining (or codetermining) the $s_i(t)$. It is, therefore, worthwhile to remember that $Y_i(t) \equiv f_i(t, \{X_j(t)\}, \Xi_i(t))$, where $y_i(t) \equiv$ realization of$(Y_i(t))$ is only one coordinate of the state $s_i(t)$ of the system, and f_i is, e.g. one of the proposed functions, maps, operations, correspondences relating the arguments of f_i to the states of the object(s). The values of the state, $y_i(t) \equiv s_i(t)$, may be model-generated, as above via f_i, or originate in empirical measurements. In the latter case, when defining the event $E_{0,i}(s_i(t)) \equiv E_{0,i}(t)$, it suffices to write, e.g. $E_{0,i}(t) \equiv (y_i(t) \in R$ or $y_i(t) \in (\overline{y}_i - \varepsilon_i(t), (\overline{y}_i + \varepsilon_i(t)))$, where $\varepsilon_i(t)$ describes some translation of a state, or $y_i(t) > 3\hat{\sigma}(y_i)$, and $\overline{y}_i$ denotes the arithmetic mean of $y_i(t)$, $t = 1, \ldots, n$, and $\hat{\sigma}(y_i)$ is the sample standard deviation of $\{y_i(t)\}$. A model-driven event $E_{0,i}$ may be defined in a similar way after replacing $E_{0,i}(t)$ by $\hat{E}_{0,i}(t) \equiv \hat{y}_i(t) \equiv \hat{f}_i(t, \{X_j(t)\}, \hat{\Xi}(t))$, where $\hat{}$ denotes an estimator of the corresponding parameter based on empirical measurements.

D3. By a process connected with the object $Ob_i(t), t \in T$ being studied or discussed, we understand a sequence of such events $\{E_{0,i}(t), t \in T\}$ that each two successive events, e.g. $(E_{0,1}(t_1), E_{0,2}(t_2), t_1 < t_2)$, $t_1, t_2 \in T$, share at least one common bearer feature of Ob_i, i.e. at least one common part of the studied object Ob_i, that has the power of preserving or changing the form or the internal structure of the object.

Remark 1.4. Definition D3, in its essence, gives a condition for a process to arise and exist during the period T. It is strictly connected to definitions D1 and D2.

D4. By a phenomenon linked to the object $Ob_i(t, t \in T)$, we understand an existing (specific or general) feature(s) that is (or may be) attributed (ascribed) to $Ob_i(t), t \in T$ in period T, place $pl(t)$ and well determined situation(s). A phenomenon make take the form of event(s), process(es).

Remark 1.5. The object Ob_i or objects $(Ob_i, Ob_j, j \in J)$ are bearers of a phenomenon via the feature(s) f_i, f_j, $j \in J$ ascribed to the object(s).

D5. By a fact (in Greek: ὅτι), we understand the determined state s_i of the object Ob_i, or event(s), process(es), phenomenon (phenomena) connected with the object Ob_i, $\{Ob_j\}$ within the period T, at position pl_i, or $\{pl_j, j \in J\}$, and in the considered research situation.

Remark 1.6. In the case of the subject of economics, by a thing (object, system) we may understand, e.g.: an economy (economies), household(s), firm(s), corporation(s), company(ies), bank(s), insurance company(ies), investment funds, stock exchanges, central bank(s), trust(s), local and central goverment(s), monopoly(ies), oligopoly(ies), consumer(s), investor(s), saver(s), union(s) of countries, economic sectors and branches, professional groups, the human population, energy resources, land, crops, production of goods, GDP, GNP, investment, consumption, balance of trade, balance of payments, wages, profits, amortization, incomes, gross (or net) inventories, demand for goods, supply of goods, employment, unemployment, prices, capital (physical, financial, human, social, intellectual, cultural), productivity of labor (or capital), innovation(s), competition, enterprennership.

Remark 1.7. Denoting the state of household h_i by $s_i \equiv s(h_i)$ at the moment of time t_a and place $pl_i(t_{a,i})$, we may write $s_i(t_a, pl_i(t_{a,i})) \equiv s_i(t_a)$ as follows:

$$s_i(t_a) \equiv (\text{size}_{h_i}(t_a), \vec{\mathbf{k}}_{h_i}(t_a), y^{disp}_{h_i}(t_a), y^{disp}_{h_i}(t_a) - \overline{y}^{disp}_{h_i}(t_a), \text{debt}_{h_i}(t_a)),$$

where: size denotes the number of people in h_i; $\vec{\mathbf{k}}_{h_i}(t_a)$ denotes a real-valued row vector with the components (coordinates) $k^{ph}_{h_i}$, $k^{f}_{h_i}$, $k^{h}_{h_i}$, $k^{s}_{h_i}$, $k^{int}_{h_i}$, $k^{cult}_{h_i}$ denoting, appropriately, the numerical monetary value of the household's physical, financial, human, social, intelectual and cultural capital; $y^{disp}_{h_i}$ denotes the current disposable income of h_i, $\overline{y}^{disp}_{h_i}$ denotes the average disposable income in the set of all $\{h_i\}$, debt_{h_i} denotes the value of the debts incurred by h_i. In order to make comparisons with other households, $\{h_j, j \neq i, j \in J\}$, possible, it suffices to normalize each component of the defined $s_i(t_a)$, either by using the appropriate means for the entire period T for each $t_a \in T$, or using the mean calculated for all the households $\{h_i\}$ at a given time t_a. The first set of means will vary according to household, the second set through time.

Instead of the mean, when suitable, it is possible and empirically valid to use the modal value of y, which is conceptually similar to Lebesgue's point of a function f, which fulfills certain assumptions formulated by Lebesgue.

Before presenting our reconstructions of the definitions of "cause", it will be informative to briefly sketch the usefulness of the concepts defined by D1–D5 within the framework of the origin of the problem of causality. For the chain $Ob \to f(Ob) \to E(f(Ob)) \to \{E(f(Ob))\}$, where Ob, f, E respectively denote: object, feature of object, event involving object, process involving object, in order to make it more causal-oriented, it is necessary to define at time $t \in T$, the spatial location $pl(Ob)$ of the object Ob, range of: t, $pl(Ob)$, f, E, $\{E\}$.

Moreover, the above comments also concern facts and phenomena.

Formulations of features should be the results of answering the questions as to why, how, in what way Ob, $f(Ob)$, $E(f(Ob))$ and $\{E(f(Ob))\}$ or changes in them are such as they are, and whether facts, phenomena and the elements of the discussed chain have one $\alpha\rho\chi\acute{\eta}$ or not (that is from what did Ob originated and in what way did it happen). An answer to the question "from what?" needs specification of the matter (substance) of Ob, and an answer to the question of "in what way?" requires specification of the (physical or abstract) properties of Ob. The first is connected with the specification of natural causes, and the latter with hypothetical causes that are conjectured, sagacious and take the form of, e.g. the properties of forces, or processes as an effect of the action of forces. This means that natural causes are considered and selected within a materialistic understanding of causality, and hypothetical causes are discussed within a phenomenalistic apprehension of causality, according to which, not substances, individuals or things, but phenomenological processes are the causes. Based on such views concerning causes, and effects treated as subsequent causes, we may distinguish two main approaches to the relation between cause and effect. The first approach, called realistic, usually corresponds to a materialistic understanding of causality. It assumes that there exists a real, objective relationship between cause and effect resulting from the action of real, objective forces of Nature emanating from Natural objects or forces.

The second approach, in its radical formulation, assumes that a causal relation is not a real, substantial, objective relation (or connection) between real substantial things (objects, systems), real processes or phenomena, but a relation between our mental representations of things in the form of ideas (concepts, notions) of things (objects). This approach is, therefore, called idealistic. There are views that the ancient opposition between Heraclit and Parmenides is still alive now. According to some, Heraclit saw a unifying and ordered principle for changes in the real world in the existence of specific laws of randomization ($\varepsilon\acute{\iota}\mu\alpha\rho\mu\acute{\varepsilon}\nu\eta$), order ($\delta\acute{\iota}\kappa\eta$) and Reason ($\lambda\acute{o}\gamma o\varsigma$). This principle enables formulating general laws for things arising and disappearing, the processes of Nature and the causes of facts, phenomena, events, processes of Nature and especially things. According to Parmenide, the inference of causes of effects manifested in changes should be carried out by reasoning based on concepts and ideas, rather than on empirical observations by the senses where someone is searching for variations in the states of things, causes and effects. Further modifications of these two scientific paradigms were formulated by Empedokles, Anaxagoras, Leukippos and Democrite. According to them, matter is composed of very small material elements called, in the case of the two last thinkers, atoms that by ever changing motions of connecting and disconnecting create new things, systems and objects. Their spatial position and motion parameters are possible and factual causes of observed effects, and there is no room for the random spatial displacements of atoms. This means that the laws determining the positions of atoms and changes in the form and internal structure of things are deterministic. For Democrite, the aim of cognition is to search for the material causes of observed phenomena. Both he and Plato saw the source of humans' "illusions" about causes in the form they use to represent reality. Concepts of causality were systemized by Aristotle. He distinguishes four kinds of causes: material, formal, purposeful and effectual. Chrissipe makes a distinction between internal general (connected with the essence of a thing) causes and external causes (acting within the environment of a thing). The ancient sceptic approach to causality was summarized by Sextus Empiricus. This approach includes negating a materialistic understanding of cause and accepting the thesis that relations between things are only mental, surmised and hypothetical.

Hereafter, we will use the symbol $R^c \equiv C$ to signify a causal relation connecting cause x and effect y. This means that the expression 'xCy' should be read as 'x causes y'. According to a modern understanding of Aristotle's thought, $\forall(x, y) : (xCy \rightarrow \neg yCx)$, thus his causal relation is asymetric. In reference to the time dimension, he states that cause x and effect y are non-simultaneous if cause x is "in potency" with respect to the effect y, although they occur simultaneously if the cause x is "in actuality" with respect to the effect y.

In Newton's view, a causal relation is determined by the causes of 'true and relative motions' of bodies, where the appropriate causes are 'the forces imposed upon bodies to generate motion'. Newton, Lagrange, Laplace, Hamilton and Maxwell formulated relations according to which the cause $x(t_a)$ was interpreted as the initial state of a thing (body) measured at time t_a and the effect $y(t_a + \tau)$ was interpreted as a later state of the body, called y, so a causal relation takes the following form '$x(t_a) \rightarrow y(t_a + \tau)$' or '$x(t_a)Cy(t_a + \tau)$'.

Hume's view is similar to Newton's with some change in the interpretation of symbols. Thus, for Hume, x denotes an event, and y another event, and C is a necessary causal connection. However, this connection is difficult to discover, due to humans' customs and habits, and the limitations of induction.

Kant accepts Hume's necessity of causation, but expresses doubts about the objectivity of perceiving objects and phenomena in the real world and the objectivity of conceptualization in the form of ordered predicates embodying the permanence of the events x, y. Hence, he concludes that the causality relation C is a priori based on human experience.

An interesting critique of Kant's approach was carried out by H. von Helmholtz, who worked successfully in physics, physiology and geometry. He was convinced that humans' spatial intuition, due to physiological limitations on the abilities of organs, is not given a priori, but forms a posteriori according to experience. However, he accepts Kant's requirements for ordering conceptual and sensual perceptions about the state of a thing, while seeking the causes of natural phenomena or processes. Laplace's concept that all physical phenomena are, in the end, reducible to the instantaneous, continuous motions of particles in space, seems to be attractive to economists if by a particle we understand, e.g., a firm, household, bank, person, corporation, consumer, or investor, when their states are determined strictly by their direct environment, which can be represented in an accurate manner.

All these thinkers or scientists acknowledged a generally understood principle of causality, which expresses the conviction that in our physical world there exists constant order. This exists independently of cognizant observers – investigators. However, for researchers, this means that there exists the possibility of researching into and discovering manifestations of the causes of the states of real objects, events, processes, facts and phenomena, as well as confirming that nihil sine causa or that similar causes or circumstances correspond to similar effects or realizations of processes. Thus, in Nature, there are no isolated phenomena, events or processes. However, the transitiveness of the similarity property of causes and effects is, in general, observed for non-living macroscopic objects (Maxwell's rule), but is not a general property of living organisms.

From this principle of causality, when dealing with causal relationships or laws, one should strictly distinguish whether the effects or causes are things (objects, systems, bodies), their states, events, processes or connected phenomena. It should be said that there are scientists and non-scientists who fundamentally disagree with the concept of causality (for example, E. Mach and his followers). However, as has been shown by B. Gawecki, even in physics, if one ignores reversible physical phenomena and uses M. Planck's criterion for differential equations describing non-reversible phenomena, then the concepts of cause and effect cannot be replaced by the concept of an ordinary mathematical function as E. Mach would like.

An economist (similar to a natural scientist) should not concentrate on a single fact, but should mainly consider regularly repeating facts, events, phenomena and processes, or analyse general features of economic phenomena that induce repetition where economic facts and phenomena occur in direct succession. Empirically and practically, it is possible to agree with J. S. Mill, since it is possible to assume direct succession if we use instant units of time (parts of seconds). Assuming simultaneous occurrences of economic facts, events and phenomena, longer units of time may be used (hours, days, weeks, months, years).These distinctions depend on the types of economic facts or phenomena. Thus, Mach and Russell's views are mathematically understandable within the continuity of arguments framework, but in practice are discussable and weak, not only in the sense of A. Meinong, C. Ducasse, Z. Zawirski and B. Gawecki, but also according to the views of C. Sigwart and M. Wartenbery stressing the importance of the feature of action within causal relations.

It should be underlined that there have been constant references by philosophers, natural sciences representers, physicians, logicians, mathematicians, engineers, sociologists, astronomers, artists, businessmen and politicians to these concepts and ideas of cause, effect and causality.

There are many definitions of cause. Below, the author gives his interpretations of some of them (see W. Milo (2011)). These definitions have the following form:

DC.1 The cause of a given effect on a thing (object, body, system), its state, an event, process, fact, phenomenon, or conclusion (consequence) of reasoning (based on given premise) is another thing(s), states of thing(s), event(s), process(es), fact(s), phenomenon (phenomena) or conclusion(s) in the form of premises, lemmas, theorems or valid sentences which evoke the effect.

DC.2 By the material (substantive) cause of a given thing (res) R_j, we understand the material, substantial part of R_j from which R_j originated and still exists, where, e.g. R_j denotes a sculpture, and the material part is bronze.

DC.3 By the formal cause of an effect on the form of a given thing (res) R_j, we understand the corresponding form(s), pattern(s), or essence, definition(s), axioms, theorems, conceptial-essential preforms. If the effect takes the form of conclusion(s) or consequence(s), then the cause is a premise(s).

DC.4 By the effectual cause of the thing R_j, a fact, state of R_j, event, phenomenon, process, conclusion(s), consequence(s), sequence(s) and theorem(s), we understand the corresponding thing(s), fact(s), states of thing(s), event(s), phenomenon (phenomena), process(es), premise(s), axiom(s) or theorem(s) that evoke the effect.

DC.5 By the final (purposeful) cause of the formation or existence of thing(s), fact(s), event(s), phenomenon (phenomena), process(es), states of thing(s), conclusion(s), consequence(s), succession(s), theorem(s) or model(s), we understand other things (objects), fact(s), states of thing(s), event(s), process(es), or premise(s), axiom(s), theorem(s), model(s) and theory(ies) argued by scientists or humans in general to evoke the purposeful goals of Nature's actions, e.g. related to thing(s), transformation(s) and, in particular, living beings, or as religious people say the aims of God.

DC.6 The cause of the existence of the thing R_j is Nature. The essence $ess(R_k)$ of the thing R_k, where $R_j \neq R_k$, satisfies the condition that $ess(R_k)$ is common to $ess(R_j)$.

DC.7 By the motional cause of the thing R_j or $\{R_j\}$, we understand the forces of pulling, pushing, stressing, disconnecting, stretching, motion, projecting, resistance, attraction and repulsion, that govern the interaction between objects of a material, both non-living and living nature on Earth, as well as notions of objects in the Universe as a whole.

DC.8 By the cause of an effect(s), given in the form of an event(s) or phenomenon (phenomena), we understand another event(s) or phenomenon (phenomena), where all the defined events, phenomena take place in well established periods of time, locations, neighboring events, phenomena and situational contexts.

DC.9 By the necessary, non-accidential cause of an effect taking the form of thing(s), event(s) or phenomenon (phenomena), we understand another thing(s), event(s), phenomenon (phenomena), which evokes the effect, where the cause and effect are well determined in time, space and contextual situations.

DC.10 By the accidental (random, incidental, chanceful, casual) cause of an effect taking the form of thing(s), event(s), phenomenon (phenomena), process(es) and fact(s), we understood another thing(s), event(s), phenomenon(s), process(es) and fact(s), which evokes the effect, where all things, events, phenomena, processes or facts take place in well defined time periods, locations, situational contexts, forms and situations that were neither known nor previously experienced. Hence, the actual or potential results of the cause are unknown, as are also unknown the scale of future interactions between future effects and present effects.

DC.11 a) By exogenous effectual and final causes, we understand such causes that generate the motion of thing(s), changes in facts and phenomena.
b) By internal causes, we understand material and formal causes.

DC.12 By the qualitative cause of an effect taking the form of changes in things or phenomena, we understand something that changes the structure of the positions of the components of these things, phenomena or changes the substantial structure of these things and phenomena.

DC.13 By the single (individual, singular, unit) cause of an effect, we understand the only origin (material, formal, effectual or final) of – in a given place, time and situation – the circumstances of the given thing(s), fact(s), event(s), phenomenon (phenomena) or process(es). By a plural (collective, holistic) cause, we understand the set (collection) of jointly interacting factors (synergeticly, non-synergeticly, in time and space) that influence the given thing(s), fact(s), event(s) or phenomenon (phenomena).

DC.14 By separated (disconnected) causes of effects, we understand such causes that cannot simultaneously evoke effects at the same place, where the causes and effects take the form of things, phenomena, processes or facts.

DC.15 The common cause of a conclusion (consequence) of reasoning is the middle term of a syllogism, where the middle and marginal terms are of the same kind with respect to time, place and quality.

DC.16 By a commensurable cause, we understand a cause that evokes a commensurable effect.

DC.17 By an aplostic (abso lute, necessary, non-sophistic, unconditional) cause, we understand a real, recognized cause which is empirically or syllogically justified or proved by many scientists using various empirical data sets and counter-factual argumentation.

DC.18 By a generic or kindful cause, we understand the gender or kind of things, events, processes, facts, phenomena, theorems or theories that determine the effect on the form of genders or kinds of the same or other things, events, phenomena, facts, processes, theorems or theories.

It is easy to check that in these 18 definitions the notion of thing was used 14 times (in all except DC.8, DC.15, DC.16, DC.17). The notion of fact was used 7 times, the notion of event was used 9 times, phenomena 9 times and process 7 times. Usually objects that play the role of causes belong to the same category of objects as the objects affected, although the particular objects of cause and effect must be different, in order to exclude the self-causation of non-living objects.

The content of the above definitions of cause may be summarized as below. Thus cause may denote:

- motive, reason, impulse, stimulus to act or decide,
- chance, condition, circumstance, situation, occasion, probability,
- factor, determinator, originator, doer,
- being, matter, substance, thing, structure of matter, material,
- state of a thing, event, phenomenon, process, fact,
- form of a thing (object), concept of a thing, essence of a thing, pattern of a thing,
- pronouncement, sentence, definition, axiom, theorem, theory, rationale, succession, consequence, predicate, proposition, judgement,
- aim of an action and interaction or aim of the existence of non-living and living beings,
- God, demiurge,
- forces of Nature (and its actions), laws of Nature, order of Nature,
- ability of things, objects, animals, human to move things,
- classes of things, objects, systems, facts, events processes, phenomena,
- synergy or asynergy of interactions between a cause or group of causes and another cause or group of causes,

- essential or accidental attributes (properties, features) of things (objects, systems), facts, phenomena, events, processes,
- quantities or qualities of things, objects, systems, bodies.

Remark 1.8. Since a cause may be singular or plural (holistic), in the above summary we may replace the singular form, e.g. motive, chance, by their plural forms, i.e. motives, chances. Moreover, it is possible to extend the list of possible definitions of the concept of the cause.

It is important to note that the concept of causality requires reference to the concept of "interaction" and the concept of an "interaction relation". The two latter notions are, in turn, connected with the concept of "passing something" from one thing (object, body, system, subject) to another thing (object, body, system, subject). Such passing may be direct (when bodies or systems are in direct contact) or indirect (when bodies or systems are not in direct contact). Since the XVII c., the dominant position in science stresses the interpretation of interaction as a thing passing from one state to another state, or passing from one event to another event, from one process to another process, from one fact to another fact, or from one phenomenon to another phenomenon, as well as mixed configurations of passing, e.g. from fact to event (or from event to phenomenon). Generally speaking, we may talk about passing from one studied object to another studied object.

A question arises as to what is this something that is passing? The following are popular names for it: "the carrier of an interaction", "the carrier of cause", "the carrier of a causal interaction". It should be stressed that interactions may be efficient (succesful, creative, effect-producing) or inefficient. It is further assumed that each interaction is causal and some causes are effective (evoke effects) and other causes are ineffective. Moreover, the degree of efficiency of an interaction may depend on the structure of the conditions necessary and sufficient for efficiency.

In order to discuss the essence of the concept of a causal interaction relation, denoted by R^c, the following notation will be used:

S^c – the thing (object, system, subject, body) that is the original cause of an indirect interaction with S^{ce}, a separate object being the cause-carrier of changes in the third, affected object S^e, where $S^c \cap S^e = \emptyset$ and $S^c \neq S^e$ indicate a lack of direct contact between S^c and S^e. There is a direct effect of S^c on S^e – the affected object, when the objects S^c and S^e are in direct contingency (contact, connection),

S^{ce} – the cause-carrier (a thing, object, body, system, subject) is an object distinct from S^c and S^e when $S^c \neq S^{ce} \neq S^e$. When S^c and S^e are overlapping, then S^{ce} is the common part of S^c and S^e

S^e, $S^e \neq S^c$ is the effect of cause S^c. It a distinct thing (object, system, subject) or different in form and structure from S^c,

T^c is the period over which the cause-object S^c exists, where T^c contains, among other things, such elements as t_0^c – the time of the origin (coming into existence) of the cause S^c, t_d^c time of disappearance of S^c, $t_{0,in}^c$ time of the initiation of causal interaction, $t_{d,in}^c$ time of the cessation of interaction,

T^{ce} the period over which the causal interaction carrier-object S^{ce} exists, which contains, among other things, such elements as t_0^{ce} – time of the origin of the carrier object S^{ce}, t_d^{ce} – time of the disappearance of the carrier S^{ce}, the time $t_{0,in}^{ce}$ of the initiation of causal interaction by S^{ce}, $t_{d,in}^{ce}$ time of the cessation of causal interaction by S_{ce},

T^e period over which the affected object (subject, body, system, thing) exists, which contains, among other things, such elements as t_0^e – time of the origin of object S^e, t_d^e – time of the disappearance of S^e, $t_{0,in}^e$ time of the initiation of causal effects on S^e, $t_{d,in}^e$ – time of the cessation of causal effects on S^e.

Remark 1.9. Both for convenience and ontological reasons we treat S^c, S^{ce}, S^e, T^c, T^{ce}, T^e as sets (in the sense of set theory) with well defined elements that have appropriate interpretations within particular fields of academic study (such as physics, chemistry, biology, economics, psychology, sociology, mathematics, statistics, econometrics, computer science, engineering, medicine and the arts). Thus, the cause S^c may be treated holistically as a single unit, object, system, subject, thing or the union, sum of parts (subsets $S_i^c, i = \overline{1, p_c}$) of a given body, object, system, ..., thing. The cause S^c may also be treated as a set (union) of single unit elements $S^c = \{s_{ij}^c : s_{ij}^c \in S_i^c, S_i \subset S^c, i = \overline{1, p_c}, j = \overline{1, q_c}\}$.

Remark 1.10. As in Remark 1.9, we may express the definitions given above using set notation, as follows:

$$S^{ce} \equiv \bigcup_i S_i^{ce}, S^{ce} \equiv \{s_{ij}^{ce} : s_{ij}^{ce} \in S_i^{ce}, S_i^{ce} \subset S^c, i = \overline{1, p_{ce}}, j = \overline{1, q_{ce}}\},$$

$$S^e \equiv \bigcup_i S_i^e, S^e \equiv \{s_{ij}^e : s_{ij}^e \in S_i^e, S_i^e \subset S^e, i = \overline{1, p_e}, j = \overline{1, q_e}\},$$

$$T^c \equiv \bigcup_i T_i^c, T^c \equiv \{t_{ij}^c : t_{ij}^c \in T_i^c, T_i^c \subset T^c, i = \overline{1, p_c^*}, j = \overline{1, q_c^*}\},$$

$$T^{ce} \equiv \bigcup_i T_i^{ce}, T^{ce} \equiv \{t_{ij}^{ce} : t_{ij}^{ce} \in T_i^{ce}, T_i^{ce} \subset T^{ce}, i = \overline{1, p_{ce}^*}, j = \overline{1, q_{ce}^*}\},$$

$$T^e \equiv \bigcup_i T_i^e, T^e \equiv \{t_{ij}^e : t_{ij}^e \in T_i^e, T_i^e \subset T^e, i = \overline{1, p_e^*}, j = \overline{1, q_e^*}\}.$$

It is easy to conjuncture that the above characterizations of S^c, S^{ce}, S^e, T^c, T^{ce}, T^e enable us to use models for the analysis of panel data and methods for testing hypotheses regarding whether $S^c, S^{ce}, S_i^c, S_i^{ce}, \forall_i$, are or are not causes of effects in the form of new $S^e, \{S_i^e\}$ or changes in the old form of $S^e, \{S_i^e\}$.

Since we are using set theory notion, we may discuss the concept of causal interaction relations or, in other words, the concept of the causality relation R^c.

The Cartesian products of the pairs of sets $(S^c, S^e), (S^{ce}, S^e), (S^c, S^{ce})$ are denoted $S^c \times S^e, S^{ce} \times S^e, S^c \times S^{ce}$, respectively, where

1. $S^c \times S^e \equiv \{< s_i^c, s_j^e >: s_i^c \in S^c \wedge s_j^e \in S^e; i = \overline{1, p_c}, j = \overline{1, q_e}\}$,

2. $S^{ce} \times S^e \equiv \{< s_i^{ce}, s_j^e >: s_i^{ce} \in S^{ce} \wedge s_j^e \in S^e; i = \overline{1, p_{ce}}, j = \overline{1, q_e}\}$,

3. $S^c \times S^{ce} \equiv \{< s_i^c, s_j^{ce} >: s_i^c \in S^c \wedge s_j^{ce} \in S^{ce}; i = \overline{1, p_c}, j = \overline{1, q_{ce}}\}$,

where the ordered pairs $< s_i^c, s_j^e >, < s_i^{ce}, s_j^e >, < s_i^c, s_j^{ce} >$ are understood in the sense of Kuratowski (1921), i.e. dropping the lower indices (i, j), by $< s^c, s^e >$ we will understand the set $\{\{s^c\}, \{s^c, s^e\}\}$ where $\{s^c, s^e\}$ denotes the set consisting of an unordered pair, and $\{s^c\} \equiv \{s^c, s^c\}$. Instead of $< x, y > \in R^c$, we will hereafter use the expression xR^cy, signifying that the cause x is in the causality relation R^c with respect to the effect y. Thus we may say that R^c is a subset of either the set $S^c \times S^e$, the set $S^{ce} \times S^e$ or the set $S^c \times S^{ce}$, i.e. that the binary causality relation may take the form $R^{c,1}$, $R^{c,2}$ or $R^{c,3}$ and $R^{c,1} \subseteq S^c \times S^e, R^{c,2} \subseteq S^{ce} \times S^e, R^{c,3} \subseteq S^c \times S^{ce}$.

By including the time sets (periods) T^c, T^{ce}, T^e within which the causality relations take place, we may define tense-object extensions of the above relations $R^{c,1}, R^{c,2}$, and $R^{c,3}$ to ternary relations, which will be denoted as:

4. $R^{c,1,t^c} \subseteq S^c \times S^e \times T^c, R^{c,2,t^c} \subseteq S^{ce} \times S^e \times T^c, R^{c,3,t^c} \subseteq S^c \times S^{ce} \times T^c$,
   or by including T^{ce} we obtain

5. $R^{c,1,t^{ce}} \subseteq S^c \times S^e \times T^{ce}, R^{c,2,t^{ce}} \subseteq S^{ce} \times S^e \times T^{ce}, R^{c,3,t^{ce}} \subseteq S^c \times S^{ce} \times T^{ce}$,
   or by including T^e we obtain

6. $R^{c,1,t^e} \subseteq S^c \times S^e \times T^e, R^{c,2,t^e} \subseteq S^{ce} \times S^e \times T^e, R^{c,3,t^e} \subseteq S^c \times S^{ce} \times T^e$.

This process of creating different kinds of causality relation R^c may be continued to include mixed Cartesian products, such as $T^c \times T^{ce}, T^c \times T^e, T^{ce} \times T^e$ and $T^c \times T^{ce} \times T^e$, so that the most detailed variant of R^c will take the form of the senary (6-ary) causality relation

7. $R^{c,md} \subseteq S^c \times S^{ce} \times S^e \times T^c \times T^{ce} \times T^e$,

where re-expressing the right hand side of this relation in terms of ordered pairs leads to $<< s^e, s^{ce}, s^e, t^c, t^{ce} >, t^e >, <<< s^e, s^{ce}, s^e, t^c >, t^{ce} >, t^e >$ and so on (for convenience we have dropped the lower indices of $\{s^\cdot\}$ and $\{t^\cdot\}$).

In the case of the binary causality relation $R^{c,i}, i = 1,2,3$, it is useful to speak about the domain $DR^{c,i}$ of $R^{c,i}$ and codomain $CD(R^{c,i})$ of $R^{c,i}$. For example, for $R^{c,1}$ we have

$$D(R^{c,1}) \equiv \{s_i^c : \exists s_j^e : (s_i^c R^{c,1} s_j^e)\},$$

$$CD(R^{c,1}) \equiv \{s_j^e : \exists s_i^c : (s_i^c R^{c,1} s_j^e)\}, i = \overline{1,p_c}, j = \overline{1,q_c}$$

The $D(R^{c,i}), \forall_i$ may also be called left-sided causality relations, and $CD(R^{c,i})$ right-sided causality relations, and their sum, $D(R^{c,i}) \cup CD(R^{c,i})$, the field of the relation $R^{c,i}$ for a given i.

Since $R^{c,1}, ..., R^{c,md}$ are sets (strictly speaking subsets of the appropriate Cartesian products of either ontic sets of things (objects, systems, subjects, bodies), sets of events, sets of processes, sets of facts or sets or phenomena), when analyzing causality, it is useful to apply well-known operations, such as summation, intersection, difference, complement of two, three, ..., n-sets. For example, in the case of binary $R^{c,1}, R^{c,2}$ we have

$$R^{c,3} \cup R^{c,2} \equiv \{< s_i^c, s_j^{ce} > \in R^{c,3} \vee < s_i^{ce}, s_j^e > \in R^{c,2}\},$$

$$R^{c,3} \cap R^{c,2} \equiv \{< s_i^c, s_j^{ce} > \in R^{c,3} \wedge < s_i^{ce}, s_j^e > \in R^{c,2}\},$$

where i, j are sets of natural numbers as in (1)–(3).

Remark 1.11. The summation operation will be purposeful when we are modeling or theorizing and wish to maximally extend the list of possible, cognitively admissible, hypothetical causes of the observed state of things, events, processes, facts or phenomena. The intersection operation may be useful when we wish to reduce the list obtained using the summation operation. The difference operation may be used to partially reduce a proposed list of hypothetical causes, and the complement operation may be useful when we want to check whether there is a joint effect of some concomitant causes.

There also exist some special relations called functional relations or functions. By the functional relation $R^f \equiv f$, we understand such a relation $R \subseteq X \times Y$ which satisfies the following two conditions:

.) $\forall x \in D(R) \exists y \in Y : (xRy)$,

..) $\forall x \in X \forall y \in Y \forall z \in Y : (xRy \wedge xRz \rightarrow y = z)$.

According to these conditions, we may say that the causal relation $R^{c,1}$ is a functional causal relation $R^{fc,1}$ if $R^{c,1} \subseteq S^c \times S^e$ satisfies the following two conditions:

...) $\forall s_i^c \in D(R^{c,1}) \exists s_j^e \in S^e : (s_i^c R^{c,1} s_j^e)$,

.iv) $\forall s_i^c \in S^c \forall s_j^e \in S^e \forall s_k^e \in S^e : (s_i^c R^{c,1} s_j^e \wedge s_i^c R^{c,1} s_k^e \rightarrow s_j^e = s_k^e)$.

Similarly, we can define the functional causal relations $R^{fc,2}, R^{fc,3}$ and more complex extensions of such relations which include the time dimension, space dimension or both.

The above presentation of causality was using the very clear and concise language of mathematical logic, as well as the more verbose languages of analytic philosophy, philosophy and economics.

It is also possible to define causality not in terms of set theory, but by a more hybrid approach that uses less formal presentation.

Below, we give some examples of such an approach.

DCIN.1 The causal object $s_i^c \in S^c$ evokes its impact on the element $s_j^e \in S^e$, which is the effect of the influence of s_i^c, due to the interaction relation R^c of the cause-carrier-element $s_k^{ce} \in S^{ce}$, where $S^{ce} \neq S^c \neq S^e$ or $S^{ce} \subseteq S^c \neq S^e$, within the period $[t_{0,in}^{ce}, t_{d,in}^{ce}]$ of interaction between the cause element s_k^{ce} in the chain $s_i^c \rightarrow s_k^{ce} \rightarrow s_j^e$ with the affected element $s_j^e \in S^e$, and where R^c is an indirect causal interaction $\forall i = \overline{1, p_c}, k = \overline{1, p_{ce}}, j = \overline{1, q_e}$.

Remark 1.12. In macrophysics, S^c might denote, e.g. the Sun, s_i^c the sun's ray and S^e the Earth or the average yearly temperature on its surface, s_j^e North America or the average yearly temperature on its surface, S^{ce} the air around the Earth, s_k^{ce} the atmosphere within 10 km of the Earth, or its average yearly temperature.

Remark 1.13. In macroeconomics, S^c might denote the world economy, s_i^c the USA economy, S^e the Canadian economy or its yearly growth rate, s_j^e Quebec's economy or its yearly growth rate, S^{ce} the USA's financial sector, s_k^{ce} - the USA subsector of investment banks or its yearly growth rate.

Remark 1.14. In mesoeconomics, S^c may represent for some fixed period of time, e.g., a given set of national investors, s_i^c a large investor, S^e the set of unemployed people of age 18–65, s_j^e the unemployed who have only elementary education, S^{ce} the set of investors investing in the service sector, s_k^{ce} mega-investor in the service sector or the yearly growth rate in their level of investment.
Remark 1.15. It should be underlined that, under various necessary and sufficient conditions for successful interactions, various carriers of causal interactions may be involved in different kinds of interactions. Definition DCIN.1 does not exclude indirect causal interactions of sequential feedback type, e.g. described as follows:

$$S^c(T_1) \to S^{ce}(T_1) \to S^e(T_1) \to S^e(T_2) \to S^{ce}(T_2) \to S^c(T_2),$$

where $T_1 \to T_2$ is the time arrow, and, e.g. $T_1 \equiv T_1^c \cap T_1^{ce} \cap T_1^e$ or $T_1 \equiv T^{ce} \cap T^e$, and the causal impact involves all factors as causes and effects in a feedback loop.
Remark 1.16. The relation R^c, in general, is not reflexive, i.e. some objects may, over a longer period, interact between themselves, others cannot. R^c is not necessarily reversible nor transitive. One can find economic interpretations of particular economic relations in terms of logical properties, for example, in H. Brems' (1968), B. Stigum (1990).
Remark 1.17. Things, objects, bodies, systems and subjects may – with respect to R^c, be connected actively, openly, or resistibly. These sorts of connections enable us to formulate more detailed and precise theses about the properties of classes of causality relations, and to mitigate some practical "weaknesses" of general formulations of the concept of a causality relation. The following are less formal, general definitions of causality:

DCIN.2 It is said that $\forall(s_i^c \in S^c, s_j^e \in S^e, t_{o,in}^{ce} \in T^{ce}, t_{d,in}^{ce} \in T^{ce}, t_{o,in}^{ce} < t_{d,in}^{ce})\exists s_k^{ce} \in S^{ce}$: where s_i^c is the ultimate cause of the effect s_j^e, s_k^{ce} is the direct cause of s_j^e and they are both causes of a change $ch(s_i^e)$ in s_i^e of either its structure or form, or they are the causes of new objects arising.

DCIN.3 It is said that $\forall(s_i^c \in S^c, s_j^e \in S^e, t_o^c, t_d^c \in T^c, t_o^e, t_d^e \in T^e, t_o^c < t_d^c, t_o^e < t_d^e, t_d^c \leq t_d^e)\exists s_i^c \in S^c$ which is a cause of $ch(s_i^e)$.

DCIN.4 It is said that $\forall(s_i^c \in S^c, s_j^e \in S^e)\exists T^c : s_i^c$ is the main cause of s_j^e.

DCIN.5 $\forall(s_i^c \in S^c, s_j^e \in S^e) : s_i^c$ is an instantenous cause of s_j^e.

DCIN.6 $\forall(s_i^c \in S^c, s_j^e \in S^e) : s_i^c R^c s_j^e$.

Remark 1.18. In definitions DCIN.3 – DCIN.6, cause-carriers and the period of their action, S^{ce}, s^{ce}, $T^{ce}, t_{o,in}^{ce}, t_{d,in}^{ce}$, are implicit. Their features may be revealed or entimematically treated as premises of causal reasoning. Definitions DCIN.1 and DCIN.2 essentially state the same fact of the existence of an ultimate cause and a direct cause with additional minor comments about time points in DCIN.2. The time dimension is omitted in DCIN.5 and DCIN.6 (in DCIN, part IN comes out from INFORMAL).

Let us recall that the sets $S^c, ..., S^e, T^c, ..., T^e$ are, by the corresponding definitions, divided into subsets (layers, panels, belts) $S^c_i, ...S^e_i, T^c_i, ..., T^e_i$ with i taking values from a set of natural numbers, i.e.

8. $S^c \equiv \cup_i S^c_i, ..., S^e \equiv \cup_i S^e_i, T^c \equiv \cup_i T^c_i, ...T^e \equiv \cup_i T^e_i$.

Note: the S^c_i and S^e_i can be interpreted as the panel subsets of causes and effects corresponding to the appropriate time interval (T^c_i or T^e_i).

By using Remarks 1.9, 1.10 and Definitions (1)-(8) we may, e.g. define the following versions of mutiple-layer causes of causal relations, i.e.

9. $R^{c,1}_i \subseteq S^c_i \times S^e_i, i = \overline{1, l_1}$;
10. $R^{c,2}_i \subseteq S^{ce}_i \times S^e_i, i = \overline{1, l_2}$;
11. $R^{c,3}_i \subseteq S^c_i \times S^{ce}_i, i = \overline{1, l_3}$;

and so on, until

12. $R^{c,md}_i \subseteq S^c_i \times ... \times S^e_i \times ... \times T^e_i, i = \overline{1, l_4}$.

For such multi-layer causal relations, based on the discussion above, we may define the domains $D(R^{c,1}_i), ..., D(R^{c,md}_i)$, and codomains $CD(R^{c,1}_i, ..., CD(R^{c,md}_i)$ for the appropriate $\{i\}$.

Thus, we may now define a multi-layer cause in the following ways (below letters L, H are first letters in "Layer" and "Holistic"):

DLC.1.: S^c_i is a multi-layer-holistic cause of a multi-layer-holistic effect S^e_i if it fulfills the condition $S^c_i R^{c,1}_i S^e_i$ or satisfies the causality relation $R^{c,1}_i : D(R^{c,1}_i) \to CD(R^{c,1}_i)$.

DLC.2.: S^c_i is a multi-layer-holistic cause of the multi-layer-holistic effect S^e_i if it fulfills $S^c_i R^{c,2}_i S^e_i$ or if $R_{c,2} : D(R^{c,2}_i) \to CD(R^{c,2}_i)$.

By continuing this process, we obtain

DLC.md: S^c_i is a multi-layer-holistic cause of the multi-layer-holistic effect S^e_i if it fulfills $S^c_i R^{c,md}_i S^e_i$ or if $R_{c,md} : D(R^{c,md}_i) \to CD(R^{c,md}_i)$.

In the totally holistic case, we may treat S^c and S^{ce} together as a combined single cause, i.e. we may define

DHC1. S^c or S^{ce} is the holistic ultimate cause or direct cause of the holistic effect S_e if they fulfill:

.) $S^cR^cS^e$ or $S^{ce}R^{ce}S^e$ or

..) $R^c : D(S^c) \to CD(S^e)$ or $R^{ce} : D(R^{ce}) \to CD(R^{ce})$.

Remark 1.19. It is important to remember that by connecting the arguments of the above causal relations $R^{c,1}, ..., R^{c,md}, R_i^{c,1}, ..., R_i^{c,md}, R^{ce}, R^c$ using suitable quantifiers, $\forall$ and $\exists$, which precede expressions such as e.g. $S^cR^cS^e$, we obtain very rich semantic interpretations. Below, we present interpretations for the general case of R^c.

We, therefore, highlight the following interpretations (meanings) of R^c:
mir1) $\exists S^c \exists S^e : S^cR^cS^e$, where $R^c : D(S^c) \to CD(S^e)$,
mir2) $\exists S^c \forall S^e : S^cR^cS^e$, $R^c : D(S^c) \to CD(S^e)$,
mir3) $\forall S^c \exists S^e : S^cR^cS^e$, $R^c : D(S^c) \to CD(S^c)$,
mir4) $\forall S^e \exists S^c : S^cR^cS^e$, $R^c : D(S^c) \to CD(S^e)$,
mir5) $\forall S^c \forall S^e : S^cR^cS^e$, $R^c : D(S^c) \to CD(S^e)$.

The interpretations of the symbolic expressions mir1-mir5 are appropriately:

mir1)

a) some holistic cause S^c has caused (generated, produced, envoked, induced) some effects S^e;

b) some holistic effect S^e was caused (generated, envoked, induced) by some causes S^c;

c) some change $ch(S^c)$ in S^c has caused (generated, produced, induced) some change $ch(S^e)$ in S^e, not excluding annihilation as a possible state of S^e .

mir2)

a) some holistic cause S^c (i.e. convolution of causes, composition of causes) is the cause of everything covered by "$\forall S^e$". This cause S^c may be interpreted as an ultimate cause or arch-cause;

b) all effects covered by "$\forall S^e$" were caused (produced, generated, induced, affected) by some singular but holistic ultimate cause S^c;

c) changes in all the effects covered by "$\exists ch(S^c) \forall S^e$" were caused (generated, induced) by some change $ch(S^c)$ in some ultimate cause S^c.

mir3)

a) the holistically treated causes S^c have evoked some holistically treated singular effect S^e;

b) some holistic effect S^e was caused (generated, evoked, induced) by the holistic causes of the form S^c;

c) a given change $ch(S^e)$ in S^e was evoked (generated, caused, induced) by the convolution of changes in the holistic causes of the form S^c.

mir4)

a) each holistic effect has its holistically treated given cause;

b) a given holistic cause S^c has evoked (generated, produced, induced) its own holistic effect for each S^e;

c) each change in a holistic effect is associated with a change in its holistic cause.

mir5)

a) each holistic cause S^c has evoked (generated, produced, induced) each holistic effect S^e and vice versa;

b) each change in every cause has evoked (generated, produced) its own change in every effect.

Remark 1.20. Similar interpretations may be formulated within the frameworks in which we consider, for each i, j, the cases of multi-layer sets $S_i^c, ...T_i^e$, and when, e.g., S^c is a set of singletons s_{ij}^c. Moreover, it should be stressed that such effects might be positive (the creation of new useful objects or desired changes in existing objects) or negative (the creation of new harmful or destructive objects or the annihilation of useful objects). The ordering of the interpretations (mir1)–(mir5) resembles an ordering of conclusions from empirical studies from particular pronouncements to general ones. It is also important to remember that the existence quantifier $\exists$ narrows the range of causality relation the most strongly when we move it to the beginning of the chain of expressions with quantifiers. In particular, this will be valid when we formulate complex definitions of R^c with added quantifiers for such arguments as $S^c, S^{ce}, S^e, T^c, ..., T^e$ taken jointly. By introducing the word "holistic" in our interpretations of causality, we want to stress our view that holistic cause and effect embrace the impact of their holistic parts, as well as particular elements of these parts (layers), but also the joint impact of elements or components of causes and effects which are in fact future causes. The latter argument is especially important for economists, biologists, physicians, politicians and military leaders.

There are, however, very important problems and topics connected with varying degrees of importance to the problems and topics concerning causality which we have omitted. The following are such topics:

p1) what are necessary and sufficient conditions for the following properties of causality relations $R^c, R^{c,1}, ..., R^{c,md}$; $R_i^{c,1}, ..., R_i^{c,md}$; $R_{ij}^{c,1}, ..., R_{ij}^{c,md}$:

  a) reflexivity, areflexivity, symmetry, asymetry,

  b) transitivity, coherence, uniqueness,

  c) ordering, weak ordering, strong ordering, identity;

p2) how can the properties in (p1) be used in the formulation of practically and theoretically admissible criteria for testing causality in particular fields of the natural and social sciences, including economics;

p3) extensions of the definitions of causality relations $R^c, ..., R_{ij}^{c,md}$ to cases where we are uncertain about S^c, S_i^c, s_{ij}^c, $S^e, S_i^e, s_{ij}^e, T^c, ..., T^e, T_i^e, t_{ij}^e$, i.e. where we treat the latter as random objects;

p4) is it possible, under (p3), to formulate verifiable theses, theories about the ontology of economic actions;

p5) can game theoretical solutions be one of the important solutions to the problems outlined in (p3, p4).

Answering and discussing the cognitive and practical importance of these problems and topics requires referencing historically acknowledged concepts, ideas, theories and models, as well as deepening their interpretations according to the current state of scientific knowledge. Also, there is a need to invent new notions and methods of scientific investigations. Keeping in mind the constant permeation of a probabilistic-statistical approach into all fields of science and scientific methodology since the XIX c., it seems that, in particular, non-unique answers will arise within the contexts of (p3)–(p5).

Leaving our discussion of (p1)–(p5) aside, we now briefly present some practically oriented problems and topics of testing for causality.

1.4. Criteria for causality

By criteria for something, we understand tests, gauges, measures, indicators or characteristic features which enable us to recognize whether something is truly what it seems to be. Here by "something" we mean the causality relation R^c discussed in Section 2. As stated there, we may distinguish between many theoretically admissible, abstract causal relations. Each of them reduces the abstract sets of causes, effects, sets of their existence, i. e. from the full sets $S^c, S^{ce}, S^e, T^c, ..., T^e$ or the appropriate layers (panels) of these sets to the appropriately determined Cartesian intersections of pairs, triads, ..., of $S^c, ..., T^e$ or $S_i^c, ..., T_i^e$ (which are subsets of the full sets or layers). Thus abstract causal relations fulfill the role of reducers or selectors of the possible unions of the original full or layered sets, where such reduction may continue until singletons form under a suitably fine decomposition of S^c to S_{ij}^e, ..., $S_{ij}^c, ...$

This decomposition (layering) process should also fulfill some reasonable criteria for qualitative and quantitative test procedures. The choice of such criteria will depend on the chosen interpretation of the elements of the original sets. If these elements are abstract objects (such as purely mathematical objects or mathematical objects with interpretations in physics, economics or engineering), then the criteria for a dependence type causality should be taken from mathematics, in particular from mathematical logic, and these should be the rules of mathematical reasoning which lead to hypotheses of truthfulness.

Thus the criteria in this case should be necessary and sufficient conditions for the truthfulness of an implication relation treated as a causality relation between the cause, understood to be the premise, and effect, understood to be the conclusion.

The considered process of selecting from relations R^c which reduces the original full sets or their layers is in fact a process of creating new abstract sets, which are Cartesian intersections of sets. Thus this process changes the structural content of the original full sets, and if carried out further on their layers, it increases, on one hand, the ease of inferring the existence and form of causality, but on the other hand, it narrows the scope for modeling the structural heterogeneity of the resulting Cartesian intersections (Cartesian products of sets), and the generality of theses or possible conjectures about the properties of causal relations (dependencies) between abstract objects (systems). One interesting question related to this selection process is whether the possible number of recognized orders and structural patterns decreases uniformly fast during this process as the number of layers (divisions of original sets) increases. Also, does there exist, for a given research problem and current knowledge, an optimal number of layers that enables achievement of the most innovative scientific results.

Thus the first general criterion for inferring a causality relation for abstract sets of objects is ccr1. The causality (dependence) relation R^c for abstract, original sets of objects is well chosen if it generates well ordered, well structured, regularly patterned codomain $CD(R^c)$ sets for qualitatively similar domain $D(R^c)$ sets. The relation R^c is perfectly chosen if it links $D(R^c)$ to $CD(R^c)$ for ordered, but not regularly patterned or structured original sets.

From a theoretical point of view, it is hard to discuss causal abstract relations without using certain fundamental concepts. These are the notions of set, relations of belonging, inclusion, identity, symmetry, monotonicity, antisymmetry, asymmetry, reflexivity, transition, order, set structure, change in the element of a set and similarity of sets. It is worthwhile to add that abstract causal relations (dependencies) are theoretical constructs (expressions) expressed either in the form of axioms or theorems, or they are formal consequences of the accepted assumptions about the attributes of the sets considered and the patterns of causal relations already recognized. This means, therefore, that criteria for causality take the forms of either logical and mathematical rules of reasoning or testing procedures for the truthfulness of conditionals, which have already, or can, be programmed in standard computer programming languages.

The use of causality in the sense of dependence was particularly strongly promoted by M. Born (1949). He does not share the view that a causal link (connection) is only the purely external succession of two physical events or states in the realization of a process. He states, instead, that a causal link is some dependence (or mutual dependence over a long time) of the effect on a given cause. He does not state clearly whether this dependence is only existence dependence, or whether it is a grounding endowment dependence relation consisting of passing on the changes arising in the structure of the causal object from the causal object to the affected object and wherever this passing on generates changes in the structure of the affected object and possibly induces replacement of the whole of S^e in the space of affected objects or, obviously, in the space of all objects considered.

Born distinguishes between two general concepts of cause and effect. The first concept assumes that causes and effects are events placed at a unique point in time and space. The second concept assumes that cause and effect are arguments of a timeless and spaceless relation between two situations belonging to two different classes of situations. The latter relation is an abstract mathematical functional relation, which has both mathematical and logical interpretations. Thus for Born, a causal link is a functional dependence between all abstract things of one kind and all abstract things of another kind. This dependence may be corroborated experimentally by doing repetitions of experiments. The practical sense of these concepts consists of considering the connection between two classes of situations for which scientists should use two different principles for testing: the principle of the antecedence of events (or situations) (when cause antecedes effect in time or space), and the principle of the contiguity of cause and effect.

If scientists are not in a position to deduce causal laws, either directly from the results of experiments or from some empirical statistical relationships called Born's statistical laws, then they should carry out further replications of an experiment in a systematic manner with suitable variations in the structure of the causal factors tested. The fact that Newton's gravitation law is deducible from Keppler's empirical astronomical relationships is of instructive importance to Born's laws.

Remark 1.21. Born's approach to causality is, in the case of deterministic causal laws, generally clear. In the case of chance (random) events, it is less clear. He does not define a chance event and does not go into a detailed argument as to why statistical laws (dependencies) are necessary and sufficient premises for selecting valid causes of effects given e.g. in the form of dependent variables in statistical or probabilistic equations. Difficulties arise due to a lack of knowledge of the true distributions of the observed objects treated as being random, existence of errors in the observations made on these objects, existence of errors in the estimation of the parameters describing statistical relations, existence of test procedures that are used in selecting the form of causality. In theoretical modern physics, scientists use mainly formal logical and mathematical criteria for causality understood as a dependency in the form of functions, functionals, operators (deterministic or stochastic), i.e. maps of their domains into (or on) their ranges (codomains). Sometimes, readers have to read a text carefully to gain the essence of what authors mean by criteria for causality (see, e.g. the paper of F. Krotscheck, W. Kundt (1978)).

Remark 1.22. It should be underlined that scientists are in a position to propose cc (criteria for causality) according to either the kinds of notions of cause described above (see the definitions of causes and their discussion given above). The lists of cc-test procedures which may be formed on the basis of these definitions, in many cases, will have different entities. It is also possible to formulate other categorizations of cc-test procedures according to particular disciplines of science, aims of scientific research, research methodologies used within various fields of science or different forms of causality relation R^c proposed by scientists.

Besides various kinds of cc implicitly mentioned in Remark 1.21, and earlier in this chapter, we may also distinguish between some more specific groups of cc-test procedures, where the distinction is carried out according to the theoretically and practically important properties of the causal relations. These properties can be divided into the following groups:

g1) logical:
non-contradiction, connectivity, asymmetry, irreflexivity, ordering, partial ordering, linear ordering, transitivity, belonging, inclusion, consequence, reversity, antisymmetry, symmetry, reflexivity, identity, uniqueness, equivalence, binarity, totality, conjugacy;

g2) mathematical, mathematical physics, engineering, economics:
continuity, regularity, monotonicity, discontinuity, periodicity, complexity, incidentality, coincidentality, boundedness, thresholding, splininess, distrubutiveness, randomness of functions as causality relations;

g3) natural, social and economic:
temporal, spatial or temporal-spatial ordering; objectivity of causal ordering, limitability of causal propagation, durability of causal propagation, robustness or sensitivity of causal propagation, necessity or sufficiency of causal propagation for global changes;

g4) medical, ecological, social, economic (where, in practice, causality takes the form of associations):
spatial and temporal consistency, coherence, strength, intensity, generality, specificity, completeness, gradiential change in bio-socio-eco components, plausibility, analogy, experimental similarity to R^c – forms of associations, overdetermination or underdetermination of causes, preemption or redundancy during cc-test procedures.

For cc-tests based on (g1)-(g4), it would be interesting to check the validity of the following conjectures:

c1) For each property mentioned in (g1)–(g4), it is possible to formulate a reasonably good cc-test procedure;

c2) Finding or constructing a concrete form of R^c which will satisfy all the properties belonging to a given group of properties and formulating a good cc-test for its validation will be extremely difficult, if not impossible, and practically unreasonable;

c3) a practical strategy of searching for good cc-procedures should be confined to relations R^c with the most important properties chosen from the point of view of the aims of a research project, as well as the available financial, technical and human resources.

Remark 1.23. Selecting plausible candidates for good tests for statistical associations and acceptable tests for causal relations requires workable statistical test procedures based on formal test statistics. Moreover, even the most commonly used tests of significance (e.g. Student's t-statistic) only give statistical conclusions regarding the significance of the influence of a chosen (variable) cause with some arbitrary chosen level of significance under the probabilistic assumptions of the model's data generating process, which should be additionally tested under the assumptions of that testing procedure (and so, until no assumptions are made by a test procedure). This hierarchy of sequential tests leads to uncertainty about how to formally choose well-justified indicators enabling us to determine to what degree the particular cc-test criteria are satisfied by a particular ca (criteria for association) – test procedures, and whether the ca-inductive tests used are sufficient evidence for stating causality of a given form of relationship R^c. Ca-tests are usually treated by physicians, sociologists, economists and psychologists as 'screening tests' to find associations which may be treated as possibility results for a causal relation (dependence function). This approach immunizes such procedures, to some extent, against gross contradictions and the illogicalities of our empirical reasoning.

It is worthwhile to recall that inferring a causal relation from an association relation can be aided by the use of either correlation (dependence) relations, regression based models, models for longitudinal data with censoring, hazard function models, autoregression models, moving-average autoregression models, structural equation models, Bayesian models, algorithmic path models. The above model relations may be linear or non-linear, identifiable or non-identifiable, transformable or non-transformable. Practitioners in economic and social studies use correlation and regression analysis, together with other statistical models, as an attempt to disentangle causes and effects with only a faint hope that they will reach such a goal. Even experienced investigators have rather vague ideas (based on their experience and intuition) as to the relevant variables, while playing their modeling game against economic and social Nature, whose patterns of associations, dependencies and connections are always changing. They are much more uncertain as to causal orderings, hypotheses regarding the direction of causation, and hypotheses about the true functional forms of causality. They choose these approaches and forms on the basis of research experience, computational convenience, familiarity of use and ease of interpretation. The main problems encountered by economists, sociologists and social psychologists are connected with a lack of high quality data or non-experimental research. Additionally, these research circumstances introduce uncertainty about whether the reliability of a test for a causal relation can, in itself, be verified.

Remark 1.24. From the statistical, econometric and probabilistic literature and our remarks presented in previous sections, it is known that the reasoning enabling us to infer from association relations through correlation relations and regression relations to causal relations is, even in the case of abstract objects, rather a matter of definitional conventions and assumptions that never hold in practice. In practical research situations, such inference (in medical, social and economic studies) is more commonly a result of scientific habits and beliefs than unquestionable scientific truths. There is a deep need to have experience in studying ontic subjects (i.e., systems met in real life, both in the sense of interpreting both real data and the results of statistical and econometrical computations, as well as assessing the empirical validness of such analysis).

This opinion follows, among other things, from the strong practical limitations on the observation and precise measurement of quantitative features, indicators of qualitative and quantitative state of objects, events and the processes linked to them, as well as facts and phenomena connected with these objects. There is also a lack of empirically testable or complete theories that explain and forecast both regular and irregular states of things, events, phenomena and processes.

In the remarks made in this section, there have been some additional descriptions and explanations concerning different forms of relations between studied objects that are or can be treated as approximations of cause-effect relations.

To summarize, we have discussed dependency, functions, functionals, operations, and association, correlation, regression and autoregression relations. Besides these approaches, the following are also growing in popularity: fuzzy set relations, concurrency relations (with priorities and free choice), sequential and non-sequential relations.

From the point of view of computer-aided modeling, the link between concurrency relations and causality relations is especially interesting. This topic integrates the research interests of modelers, applied mathematicians, engineers, programmers, statisticians and econometricians. Many particular approaches have been developed within this field including category theory (see e.g. S. Fröschle, S. Lasota (2007)) and the concept of posets applied to concurrency theory. The first approach relates and unifies models of concurrency (useful for engineers, econometricians and physicists, when they do parallel simulations). Notions related to category, such as structure, adjunction, reflection, coreflection, morphism and treating models as categories enables a deeper recognition of relations between empirical models, and establishing so called causal trees of dependence using event tree structures. For those who study cyclic phenomena, the possibility of investigating synchronization and asynchronization trees with no cyclic behaviour would be particularly interesting. It has been found, that under given assumptions, causality models are more fundamental theoretically than truly concurrent models, because they can infer causality relations without using the notion of event. However, the latter type of models may more expressively illustrate the model's interior structure using event trees. For cyclic phenomena, which are so important practically, there are still many open questions that should be answered.

The second approach, based on concurrency theory, (see, e.g. R. Janicki, M. Koutny (1997)) focuses on inferring the properties of relational structures that have important consequences, both theoretically and practically. The cited authors base their discussion on E. Szpilrajn's theorem (1930), in which it is stated that each partial order is an intersection of its total order extensions. They base their analysis on the triadic structure $(S^c, R^{c,1}, R^{c,2})$, where $R^{c,1}$, $R^{c,2}$ are binary causal relations, $R^{c,1}$ is strong and $R^{c,2}$ is weak, and both of them are defined on the cause domain S^c. Within this structure, it is possible to formulate a model of concurrent histories of given phenomenon or phenomena which describes a partial ordering of causal patterns. It is possible to obtain more detailed descriptions of internal causal structures by extending the above triadic structure to, e.g. the structure $(S_1^c, S_2^c, R_1^{c,1}, R_1^{c,2}, R_2^{c,1}, R_2^{c,2})$, where S_1^c and S_2^c are layers of the set of causes, and the two pairs of causal relations correspond to the layers, where $(c, 1)$ and $(c, 2)$ are defined to be the set of strong and weak relations, respectively, or direct and indirect. For modelers, the second, more complex, structure seems to be more promising in practice.

Finally, it is worthwhile to underline that the most important, but difficult, problem to solve is that of choosing empirically efficient and valid tests of causality. Why this is so can be shown on the basis of C. Granger's operational test of causality (1969), which is often used by econometricians and economists. The formal definition of Granger's concept of causality can be represented as follows:

DC.G: It is said that a given stationary random process $\{X_t, t = 1, ..., t_a\} \equiv \mathbf{X}_{(t_a)}$ is causing a stationary effect process $\{Y_t, t = 1, ..., t_a\} \equiv \mathbf{Y}_{(t_a)}$, i. e. $\mathbf{X}_{(t_a)} \stackrel{R^c}{\rightarrow} \mathbf{Y}_{(t_a)}$ if the process of prediction errors $\mathbf{E}_{(t_a)}$ based on $\mathbf{E}_{(t_a)} = \mathbf{Y}_{(t_a)} - \hat{\mathbf{Y}}_{(t_a)}$ has strictly smaller conditional variance when the predictor $\hat{\mathbf{Y}}_{(t_a)}$ is calculated using all the available information $\mathbf{I}_{(t_a)}$ about all the random processes assumed to be candidate causes than the conditional variance of the process $\mathbf{E}_{(t_a)}$ computed under the condition that $\mathbf{I}_{(t_a)}$ is replaced by the difference $\mathbf{I}_{(t_a)} \setminus \mathbf{X}_{(t_a)}$.

Definition DC.G gives grounds for testing the hypothesis of the existence of a relation R^c:

*) $R^c : var(\mathbf{E}_{(t_a)}|\mathbf{I}_{(t_a)})) < var(\mathbf{E}_{(t_a)}, \mathbf{I}_{(t_a)} \setminus \mathbf{X}_{(t_a)})$, where var denotes the conditional variance of prediction errors.

Remark 1.25. The relation (*) is theoretical. It can be rewritten in the form

$$\mathbf{W}_{(t_a)} \equiv var(\mathbf{E}_{(t_a)}|\mathbf{I}_{(t_a)}) - var(\mathbf{E}_{(t_a)}|\mathbf{I}_{(t_a)} \setminus \mathbf{X}_{(t_a)}) \leq \mathbf{0},$$

where $\mathbf{E}_{(t_a)}, \mathbf{X}_{(t_a)}, \mathbf{Y}_{(t_a)}, \hat{\mathbf{Y}}_{(t_a)}$ are random stationary processes, and $\mathbf{W}_{(t_a)}$, $\mathbf{I}_{(t_a)}, \mathbf{0}$ are matrices of the appropriate dimensions.

In the case where the first set objects are scalars, then var denotes scalar variance σ^2, and for each individual time t, we have:

**) $R^c : \sigma^2(\mathbf{E}_t|\mathbf{I}_{(t)}) < \sigma^2(\mathbf{E}_t|\mathbf{I}_{(t)} \setminus \mathbf{X}_t)$, and $\mathbf{W}_t \equiv \sigma^2(\mathbf{E}_t|\mathbf{I}_{(t)}) - \sigma^2(\mathbf{E}_t|\mathbf{I}_{(t)} \setminus \mathbf{X}_t)$.

It should be noted that in the case of (**), σ^2 is determined for given analytical probability distribution, for example a continuous one. In this case, economists must decide what form to choose from the infinite set of continuous distributions. Secondly, the forms of probability distributions must be chosen for X_t, Y_t, $\hat{Y}_t$ and $(Y_t - \hat{Y}_t) \equiv E_t$, and the conditional variances $\sigma^2()$ calculated according to the chosen forms of these distributions. Thirdly, they should also select, possibly in an optimal way, the functional form of the predictor $\hat{Y}_t$ according to the chosen principles and operational criteria which must be consistent with the relevant economic theories and their models (see our chapters on theories and models). After doing this, economists must select estimators of the structural model parameters for $\hat{Y}$, and then choose estimators for σ^2, i.e. choose the appropriate formula for calculating $\hat{\sigma}^2$. Fourthly, they should, at each step, test hypotheses regarding their truthness, confidence and significance.

In the case of (*), economists would be in an even more uncomfortable research situation, since *var* denotes a dispersion matrix, which would make reasoning much more complicated when calculating the second moments of the theoretical distributions. Also, there would be huge problems regarding estimation and testing hypotheses.

Remark 1.26. The DC.G definition of causality is valid only for random stationary processes. Micro, mesoeconomic and many macroeconomic processes are, however, nonstationary. These, and above given facts and their effects are causing that statistical causality tests of Granger's type would have low empirical power. Research situation is better in the case of controlled experiments or a little better in situations close to the case of controlled experiments. In the last case methodology of statistical intervention analysis or treatment effects analysis may help to increase empirical power of parametric or nonparametric tests of statistical help to increase empirical power of parametric or nonparametric tests of statistical causality.

1.5. Final notes

We have omitted many theoretical and practical problems, as well as topics. These concern both methods of confirming truth and truthfulness, as well as methods of analyzing causality, in particular methods of inferring causal relations. Economists now have to deal with huge data sets that are, however, censored by economic institutions, units, Central Statistical Offices, National Banks or Governments.

Secondly, the following practical factors may well have important practical consequences: the censoring or omission of observations, atypical results of measurement, dividing the effects of causality into mean, variance, covariance, median and mode parts within the numerical range of an effect-variable process. It would also be worthwhile to check whether these consequences have a significant impact on the results of causality analyses, the accuracy of empirical predictions and economic achievement.

More information about the topics discussed and omitted here can be found either in the literature cited below or the literature that is cited there.

Bibliography

Aghion, P., Howitt, P. (1992). A model of growth through creative destruction. *Econometrica*, vol. 60, pp. 323–351.

Aghion, P., Howitt, P. (1998). *Endogenous growth theory.* The MIT Press, Cambridge.

Ajdukiewicz, K. (1934). Das Weltbild und die Begriffsapparatur. *Erkenntnis*, 4(1), pp. 259–287.

Albert, D. Z. (2009). *Time and Chance.* Harvard University Press, Cambridge, Mass.

Antoine, A., & Pierre, N. (1970). *La Logique ou l'art de penser.* Paris: Vrin, 1981.

Aquinas, Th. (2002). *Super librum de causis.* J. Vrin Paris.

Aristotle. *Works* (Vol. I–X), Clarendon Press. Oxford.

Arnaud, A., & Nicole, P. (1993). *La logique ou l'art de penser* [1683]. Paris: Vrin, 37.

Ashby, W. R. (1956). *An introduction to cybernetics.* Wiley New York.

Asher, H. B. (1976). *Causal modeling* (No. 1–8). Sage, Beverly Hills, Calif.

Augustinus Aurelius Saint. *Soliloquia* (in Polish). Polish Scientific Publ., Warsaw.

Augustynek, Z. (1976). Past, present and future in relativity. *Studia Logica*, 35(1), pp. 45–53.

Avicenna, Ibn Sina (2010). *Danish Nama – Księga Wiedzy* (in Polish). Polish Scientific Publ, Warsaw.

Ayer, A. J. (1941). *The foundations of empirical knowledge.* Macmillan, London.

Ayer, A. J. (2012). *Language, truth and logic.* Courier Corporation.

Bagozzi, R. P. (1980). *Causal models in marketing.* Wiley, New York.

Bailey, S. (1844). *Essays on the Pursuit of Truth and on the Progress of Knowledge.* Longman, Brown, Green, and Longmans.

Basmann, R. L. (1965). A note on the statistical testability of 'explicit causal chains' against the class of 'interdependent' models. *Journal of the American Statistical Association*, 60(312), pp. 1080–1093.

Basmann, R. L. (1988). Causality tests and observationally equivalent representations of econometric models. *Journal of Econometrics*, 39(1–2), pp. 69–104.

Bayes, T., Price, R., & Canton, J. (1763). *An essay towards solving a problem in the doctrine of chances* (pp. 370–418). C. Davis, Printer to the Royal Society of London.

Betz, G. (2015). Truth in Evidence and Truth in Arguments without Logical Omniscience. *The British Journal for the Philosophy of Science*, 67(4), pp. 1117–1137.

Bienaymé, I. J. (1840). Probabilités. Principe nouveau du calcul des probabilités avec ses application aux sciences d'observation (I.) Sur la constance des causes, conclue des effets observés (B.). *Societé Philomatique de Paris Extraits*, Ser, 5, pp. 37–43.

Blalock, H. M. (1964). *Causal inferences in nonexperimental research.* Norton, New York.

Blaug, M. (1993). *The methodology of economics.* Cambridge Books.

Bocheński, J. M. (1992). *Współczesne metody myślenia.* Poznań, W drodze.

Boehm, M. (2014). Hume's definitions of 'Cause': Without idealizations, within the bounds of science. *Synthese*, 191(16), pp. 3803–3819.

Bohm, D., & Lindsay, R. B. (1957). Causality and chance in modern physics. *Physics Today*, 10(11), pp. 30–32.

Bollen, K. A.(1989). *Structural equations with latent variables.* John Wiley. New York, pp. 612–621.

Bolzano, B. (1929). *Wissenschaftslehre* (Vol. 1). F. Meiner.

Boole, G. (1952). *Studies in logic and probability.* Watts, London.

Born, M. (1949). *Natural philosophy of cause and chance.* Clarendon Press, Oxford.

Bovens, L., & Hartmann, S. (2003). *Bayesian epistemology.* Oxford University Press, Oxford.

Breitung, J., & Candelon, B. (2006). Testing for short-and long-run causality: A frequency-domain approach. *Journal of Econometrics*, 132(2), pp. 363–378.

Brems H. (1968). *Quantitative economic theory.* J. Wiley. New York.

Brentano, F. (1974). *Wahrheit und evidenz: erkenntnistheoretische abhandlungen und Briefe* (Vol. 201). Meiner Verlag.

Bunge, M. A. (1959). *Causality.* Harvard University Press, Cambridge, Massachussets.

Burks, A. W. (1951). VIII. – The Logic of Causal Propositions. *Mind*, 60(239), pp. 363–382.

Carnap, R. (1988). *Meaning and necessity: a study in semantics and modal logic.* University of Chicago Press, Chicago.

Carnap, R. (1969). *Einführung in die Philosophie der Naturwissenschaft* (Vol. 36). Nymphenburger Verlagshandlung.

Cartwright, N., & McMullin, E. (1984). *How the laws of physics lie.* Clarendon Press, Oxford.

Cartwright, N. (2007). *Hunting causes and using them: Approaches in philosophy and economics.* Cambridge University Press, Cambridge.

Cheysson, E. (1885). Review of Galton's "Regression towards...". *LeTemps* No. 23 Oct.

Chisholm, R. (1970). *Events and propositions. Noûs*, pp. 15-24.

Chwistek, L. (1912). *Zasada sprzeczności w świetle nowszych badań Bertranda Russella.* Nakł. Akademii Umiejętności, Kraków.

Clemen, R. T., Fischer, G. W., & Winkler, R. L. (2000). Assessing dependence: Some experimental results. *Management Science*, 46(8), pp. 1100–1115.

Collins, J. (2000). Preemptive prevention. *The Journal of Philosophy*, 97(4), pp. 223–234.

Cournot, A. A. (1843). *Exposition de la théorie des chances et des probabilités.* L. Hachette, Paris.

Cox, D. R. (1992). Causality: some statistical aspects. *Journal of the Royal Statistical Society. Series A (Statistics in Society)*, pp. 291–301.

Czerwiński, Z. (1960). O pojęciu przyczyny i kanonach Milla. *Studia Logica: An International Journal for Symbolic Logic*, 9, pp. 37–62.

Czeżowski, T. (1933). *Jak powstało zagadnienie przyczynowości.* Wilno.

Davidson, D. (1967). Truth and meaning. *Synthese*, 17(1), pp. 304–323.

Davidson, D. (2001). *Essays on actions and events: Philosophical essays (Vol. 1).* Oxford University Press, Oxford.

Dempster, A. P. (1990). Causality and statistics. *Journal of Statistical Planning and Inference*, 25(3), pp. 261–278.

Descartes, R., & Simon, J. (1842). *Oeuvres de Descartes.* Charpentier.

Dowe, P. (2007). *Physical causation.* Cambridge University Press, Cambridge.

Ducasse, C. J. (2015). *Truth, knowledge and causation.* Routledge.

Dummett, M. (1978). *Truth and other enigmas.* Harvard University Press, Cambridge, Mass.

Earman, J. (1976). Causation: A matter of life and death. *The Journal of Philosophy*, pp. 5–25.

Eells, E. (1991). *Probabilistic causality (Vol. 1).* Cambridge University Press, Cambridge.

Ehring, D. (1997). *Causation and persistence: A theory of causation.* Oxford University Press, Oxford.

Fetzer, J. H. (ed.). (2012). *Probability and causality: essays in honor of Wesley C. Salmon (Vol. 192).* Springer Science & Business Media.

Finetti, De, B. (1937). La pr evision: ses lois logiques, ses sources subjectives. *Annales de l'institut Henri Poincaré*, Vol. 7, No. 1, pp. 1–68.

Frank, P. (1938). *Le principe de causalité et ses limites.* Flammaration Paris.

Frege, G. (1997). *The Frege Reader.* Blackwell, Oxford.

Frisch, M. (2014). Physics and the human face of causation. *Topoi*, 33(2), pp. 407–419.

Gawecki, B. J. (1969). *Zagadnienie przyczynowości w fizyce.* Pax, Warsaw.

Geweke, J. (1982). Measurement of linear dependence and feedback between multiple time series. *Journal of the American statistical association*, 77(378), pp. 304–313.

Glymour, C., Scheines, R., & Spirtes, P. (2014). *Discovering causal structure: Artificial intelligence, philosophy of science, and statistical modeling.* Academic Press.

Good, I. J. (1961). A causal calculus (I, II). *The British Journal for the Philosophy of Science*, 11(44), pp. 305–318, pp. 43–51.

Goldman, A. I. (1999). *Knowledge in a social world* (Vol. 281). Oxford, Clarendon Press.

Goodman, N. (1983). *Fact, fiction, and forecast.* Harvard University Press, Cambridge, Mass.

Granger, C. W. (1969). Investigating causal relations by econometric models and cross-spectral methods. *Econometrica: Journal of the Econometric Society*, pp. 424–438.

Granger, C. W. (1980). Testing for causality: a personal viewpoint. *Journal of Economic Dynamics and Control*, 2, pp. 329–352.

Greenland, S., Robins, J. M., & Pearl, J. (1999). Confounding and collapsibility in causal inference. *Statistical Science*, 29–46.

Grobler, A. (2006). *Metodologia nauk.* Wydawnictwo Aureus.

Halbach, V. (2014). *Axiomatic theories of truth.* Cambridge University Press.

Hausman, D. M. (1994). *The philosophy of economics: an anthology.* Cambridge University Press.

Heckman, J. J. (2008). Econometric causality. *International Statistical Review*, 76(1), pp. 1–27.

Heidegger, M. (2006). *Der Satz vom Grund.* Klett-Cotta.

Hendry, D. F. (1995). *Dynamic econometrics.* Oxford University Press, Oxford.

Hoover, K. D. (1990). The logic of causal inference: Econometrics and the conditional analysis of causation. *Economics & Philosophy*, 6(2), pp. 207–234.

Hoover, K. D. (2001). *Causality in macroeconomics.* Cambridge University Press.

Hume, D. (1904). *An enquiry concerning human understanding.* T. K. Paul (ed.). Open Court Publishing Company.

Ingarden, R. (1960). *Spór o istnienie świata (Vol. 2).* PWN.

Irzik, G. (1996). Can causes be reduced to correlations? *The British Journal for the Philosophy of Science*, 47(2), pp. 249–270.

Israeli, I. (IX/X a.c.). A book of definitions (in Latin).

Jadacki, J. J. (1985). Spór o granice poznania: prolegomena do epistemologii. Państwowe Wydawn. Naukowe.

James, L. R., Mulaik, S. A., & Brett, J. M. (1982). *Causal analysis: Assumptions, models, and data (Vol. 1).* SAGE Publications, Incorporated.

James, W. (1975). *Pragmatism (Vol. 1).* Harvard University Press.

Janicki, R., & Koutny, M. (1997). Fundamentals of modelling concurrency using discrete relational structures. *Acta Informatica*, 34(5), pp. 367–388.

Jaskóła, J., & Olejarczyk, A. (2003). *Prawda a metoda. Cz. 2.* Wyd. Uniw. Wrocł. Wrocław.

Jeffreys, H. (1973). *Scientific inference.* Cambridge University Press.

Jevons, W. S. (1958). Principles of Science. *Daedalus*, 87(4), pp. 148–154.

Kant, I. (1793). *Kritik der Urtheilskraft (Vol. 5).* Ripol Klassik.

Kant, I. (1801). *Logik.* Königsberg.

Kenny, D. A. (1979). *Correlation and causation.* Wiley, New York.

Keynes, J. M. (2013). *A treatise on probability.* Courier Corporation.

Kiczuk, S. (1995). *Związek przyczynowy a logika przyczynowości.* Redakcja Wydawnictw Katolickiego Uniwersytetu Lubelskiego.

Koyré, A. (1957). *From the closed world to the infinite universe (Vol. 1).* Library of Alexandria.

Krajewski, W. (1967). *Związek przyczynowy.* PWN, Warsaw.

Krotscheck, E., & Kundt, W. (1978). Causality criteria. *Communications in Mathematical Physics*, 60(2), pp. 171–180.

Kuratowski, C. (1921). Sur la notion de l'ordre dans la theorie des ensembles. *Fundamenta mathematicae*, 2(1), pp. 161–171.

Leibniz, G. W. (1960). *Die philosophischen Schriften von Gottfried Wilhelm Leibniz.* C. I. Gerhardt (ed.). Olms.

LeRoy, S. F. (1995). Causal orderings. *Macroeconometrics*, pp. 211–233. Springer, Dordrecht.

Lewis, D. (2013). *Counterfactuals.* John Wiley & Sons.

Łukasiewicz, J. (1961). *Z zagadnień logiki i filozofii: pisma wybrane.* PWN, Warsaw.

Mackie, J. L. (1965). Causes and conditions. *American Philosophical Quarterly*, 2(4), pp. 245–264.

Mandlin, T. (2004). *Truth and paradox.* Clarendon Press, Oxford.

Mari, D. D., & Kotz, S. (2001). *Correlation and Dependence.* World Scientific.

McDonald, B. E. (2000). On meaningfulness and truth. *Journal of Philosophical Logic*, 29(5), pp. 433–482.

McGee, V. (1991). *Truth, vagueness, and paradox: An essay on the logic of truth.* Hackett Publishing.

McTaggart, J. M. E., & Broad, C. D. (1988). *The nature of existence (Vol. 2).* CUP Archive.

Mehlberg, H. (1969). The problem of causality in an indeterministic science. *International Journal of Theoretical Physics*, 2(4), pp. 351–372.

Mill, J. S. (1975). *A System of Logic.* Longmans, London.

Miller, D. (1966). A paradox of information. *The British Journal for the Philosophy of Science*, 17(1), pp. 59–61.

Milo, W., & Łapińska-Sobczak, N. (2002). *Stabilność rynków finansowych a wzrost gospodarczy.* Wydaw. Uniwersytetu Łódzkiego.

Milo, W. (2011). *Prawda i przyczyna. Cz. 1.* Zeszyty Naukowe, Uniwersytet Ekonomiczny w Poznaniu, pp. 199–209.

Mizon, G. E., & Richard, J. F. (1986). The encompassing principle and its application to testing non-nested hypotheses. *Econometrica: Journal of the Econometric Society*, pp. 657–678.

Morgan, M. S. (1997). The technology of analogical models: Irving Fisher's monetary worlds. *Philosophy of Science*, 64, S304–S314.

Moseley, F. (ed.). (1995). *Heterodox Economic Theories: True Or False?* Edward Elgar Publishing.

Nagel, E., & Hawkins, D. (1961). The structure of science. *American Journal of Physics*, 29(10), pp. 716–716.

Neander, K. (1991). Functions as selected effects: The conceptual analyst's defense. *Philosophy of Science*, 58(2), pp. 168–184.

Neurath, O., Morris, C. W., & Carnap, R. (1971). *Foundations of the unity of science: toward an international encyclopedia of unified science.* University of Chicago Press.

Newton-Smith, W. H. (2002). *The rationality of science.* Routledge.

Nowaczyk, A. (1998). Czy Tarski zdefiniowałpojęcie prawdy? *Przegląd Filozoficzny*, VII, Nr 2 (26), pp. 5–30.

Nowaczyk, A. (2004). *Wprowadzenie do filozofii matematyki.* Wydaw. Uniwersytetu Łódzkiego.

Ockham, W. (2010). *Suma logiczna.* Wydawnictwo Naukowe PWN, Warszawa.

Otte, R. (1981). A critique of Suppes' theory of probabilistic causality. *Synthese*, 48(2), pp. 167–189.

Owens, D. (1992). *Causes and coincidences.* Cambridge University Press.

Papineau, D. (1985). Probabilities and causes. *The Journal of Philosophy*, 82(2), pp. 57–74.

Papineau, D. (1991). Correlations and Causes. *The British Journal for the Philosophy of Science*, 42(3), pp. 397–412.

Pearl, J. (2003). Causality: models, reasoning and inference. *Econometric Theory*, 19(675–685), 46.

Peirce, C. S. (1974). *Collected papers of Charles Sanders Peirce (Vol. 2).* Harvard University Press.

Poincaré, H. (1909). *Science et méthode.* Flammaration, Paris.

Pollock, J. L. (2001). Defeasible reasoning with variable degrees of justification. *Artificial intelligence*, 133(1–2), pp. 233–282.

Popper, K. R. (1994). *Logik der Forschung.* 10. Aufl., Tübingen, 3.

Prior, A. N. (1967). *Past, present and future (Vol. 154).* Clarendon Press, Oxford.

Putnam, H. (1981). *Reason, truth and history (Vol. 3).* Cambridge University Press.

Putnam, H. (1994). *Words and Life.* Ed. James Conant. Cambridge: Harvard University Press, 503, pp. 211–217.

Quine, W. V. O., Churchland, P. S., & Føllesdal, D. (2013). *Word and object.* MIT press.

Ramsey, F. P., & Moore, G. E. (1927). Symposium: Facts and propositions. *Proceedings of the Aristotelian Society*, Supplementary Volumes, 7, pp. 153–206.

Reichenbach, M., & Morrison, P. (1956). *The direction of time.* UCP, Berkeley, California.

Rescher, N. (1973). *The coherence theory of truth.* Oxford University Press, Oxford.

Robins, J. M. (1997). Marginal structural models. *Proceed. of the Amer. Stat. Assoc.* pp. 1–10.

Russell, B. (2013). *An inquiry into meaning and truth.* Routledge.

Salmon, W. (1984). *Scientific explanation and the causal structure of the world.* Princeton University Press, Princeton.

Shafer, G. (1996). *The art of causal conjecture.* MIT press.

Simon, H. A. (1954). Spurious correlation: a causal interpretation. *Journal of the American Statistical Association*, 49(267), pp. 467–479.

Sims, C. A. (1980). Macroeconomics and reality. *Econometrica: Journal of the Econometric Society*, pp. 1–48.

Smith, A. (1817). *An Inquiry into the Nature and Causes of the Wealth of Nations (Vol. 2).* Ripol Klassik.

Sosa, E., & Tooley, M. (1993). *Causation.* Oxford University Press, Oxford.

Spirtes, P., Glymour, C. N., & Scheines, R. (2000). Causation, prediction, and search. MIT press.

Spohn, W. (2008). *Causation, Coherence and Concepts: A Collection of Essays (Vol. 256).* Springer Science & Business Media.

Stigum B. P. (1990). *Toward a formal science of economics.* The MIT Press, Cambridge, Mass.

Suppes, P. (1970). *A probabilistic theory of causality* (pp. 7–8). Amsterdam: North-Holland Publishing Company.

Szpilrajn, E. (1930). Sur l'extension de l'ordre partiel. *Fundamenta mathematicae*, 16(1), pp. 386–389.

Szubka, T. (2003). Epistemiczna koncepcja prawdy a relatywizm. *Filozofia Nauki*, 2–3, pp. 79–99.

Szymura, J. (1995). Czy rozmowy o prawdzie są bezprzedmiotowe? *Kwartalnik Filozoficzny*, 23, pp. 27–35.

Tarski, A. (1933). *Pojęcie prawdy w językach nauk dedukcyjnych.* Prace Tow. Nauk. Warsz. T34 Warsaw.

Tarski, A. (1983). *Logic, semantics, metamathematics: papers from 1923 to 1938.* Hackett Publishing.

Tkaczyk, M. (2009). *Logika czasu empirycznego. Funktor realizacji czasowej w językach teorii fizykalnych.* Wyd. KUL, Lublin.

Tooley, M. (2009). *Causation.* Oxford University Press, Oxford.

Tuchańska, B. (2012). *Dlaczego prawda?: prawda jako wartość w sztuce, nauce i codzienności.* Wydawnictwo Poltext, Warsaw.

Twardowski, K. (1903). Über sogenannte relative Wahrheiten. *Archie f. Syst. Philos*, T8, pp. 439–447.

Tworak, Z. (2009). *Współczesne teorie prawdy.* Wydawnictwo Naukowe Uniwersytetu im. Adama Mickiewicza, Poznań.

Wallace, W. A. (1972). *Causality and scientific explanation.* T1, T2. The Univ. of Michigan Press. Ann Arbor.

Ward, A. (2013). Spurious Correlations and Causal Inferences. *Erkenntnis*, 78(3), pp. 699–712.

Whitehead, A. N. (2010). *Process and reality.* Simon and Schuster.

Woodward, J. (1995). *Causation and Explanation in Linear Models.* On the Reliability of Economic Models: Essays in the Philosophy of Economics, pp. 9–62, Kluwer Boston.

Wójcicki, R., Belnap, N. D., Steel, T. B., Kréjdlin, G. E., & Bogdan, R. J. (1983). *Wykłady z metodologii nauk.* PWN Warsaw.

Wold, H. (1954). Causality and econometrics. *Econometrica: Journal of the Econometric Society*, pp. 162–177.

Woleński, J., & Simons, P. (1989). *De veritate: Austro-Polish contributions to the theory of truth from Brentano to Tarski.* The Vienna Circle and the Lvov-Warsaw School (pp. 391–442). Kluwer. Dordrecht.

Woleński, J. (2005). *Epistemologia.* PWN, Warsaw.

Wright, von G. H. (1974). *Causality and determinism.* Columbia University Press, New York.

Zellner, A. (1979). Causality and econometrics. *Carnegie-Rochester conference series on public policy* (Vol. 10, pp. 9–54). North-Holland.

Chapter 2

On randomness

Summary

The notions of random and randomness are commonly used in many fields of science, in everyday human conversation, communication, and in descriptions of many modern media events. It is, therefore, useful and interesting to present a new discussion of the meanings of these concepts held at different stages of the development of science, together with their roles in the theoretical and applied sciences. As may be seen, these notions are very complex. Their role has been changing over time. It is also interesting to see how influential state powers and their influential agents realize their ideologies, approaches to science, religion and a society's civil laws, as well as approaches to truth. In this respect, concepts of randomness are particularly useful when exercising power behind the curtain of a society's imperfect knowledge. Our attention here is devoted mainly to present semantic meanings of accidentality, which was used in science up to the times of Kant. References to current possibilities of using these meanings in the applied sciences are given when suitable. The usefulness of these concepts can also be observed in contemporary economics. Most modern textbooks, monographies and articles use, in effect, the concepts and calculus of randomness within, e.g. the framework of the econometric and statistical analysis of economic, demographic and social data.

Keywords: random, randomness, model, modeling, testing.

2.1. Introduction

The use of the word random in the sense of a random fact, feature, event, phenomenon or process indicated and still indicates non-significance or small role and importance. Randomness, in turn, denotes a feature, a property of something being random. Accidens (in Latin) or $\sigma\upsilon\mu\beta\varepsilon\beta\eta\kappa o\varsigma$ (in Greek) in philosophy means a way of being (see. A. Maryniarczyk, 2000, p. 115–122), existence or an element or part of being or properties of being. In logic, it means sentences or predicates that state accidental features about the subject considered. One interesting metaphysical interpretation of Aristotle's concepts is that randomness denotes something real that cannot exist independently from the subject or substance of things.

As often said, the whole is not the sum of its parts, thus it is useful to say that the parts have *ens in alio*, and are dependent on the existence of given things. Randomness, however, increases the possibilities of distinguishing between things according to many categories such as, e.g. color, shape, time, place, action and reaction.

It is curious that Aristotle distinguished only one kind of significant essential, necessary non-random pronouncements about the property of a thing's existence, but nine kinds of random ones. Accidental predicates concern the properties of things that only arise accidentally or intermittently, so they arise or vanish without the subject vanishing.

The ontic status of accidentality according to Aquinus was challenged by W. Ockham and Suarez. They rejected the interpretation of substance as *ens in se* and randomness as *ens entis*. In addition, they did not treat the relation between the subject-substance and its part that is random as being necessary. Ockham rejected the reality of accidentals (qualities), since the only realities, according to him, are concrete existing beings. Suarez rejected any real difference between essence and existence, substance and accidens. These views later gave the basis for dualism and scientism. Descartes considered particular changing qualities and accidens in "modi" as a product of Reason. Leibniz, in turn, considered accidens from the viewpoint of changes in and the potentiality of substance, as possibility in the form of *vives agenda* (the force of action) held by a subject-substance. According to Locke, accidens is the property of being able to create in our minds the ideas that are components of the idea of substance.

Hume wrote that concepts, in particular randomness as a concept, are subjective creations of Reason, memories and habits, but are not really existing objects. According to Kant, accidens is a derivative of Reason in the form of cognitive *modi* acts. However, Parmenides, Heraclitus, Plato and

Aristotle had already treated *variabilite, generatio et corruptio*, and *steresis*, as symptoms of the *contingentia* of things, but not as having the reason of their existence in their being itself.

2.2. Qualitative concepts of randomness from antiquity until the XIX c.

In modern science we link the word "random" to, e.g., observation, experiment, sample, event, variable, parameter, process, phenomenon, data, equation, model, system, loss function, set, action, interaction, family, map, mapping, transformation, function, functional, population, number, generator, field, vector, matrix, walk and space.

Intuitively, many of these links are understood as connections that happened, happen or will happen without any expected, believed, definitely justified pattern, aim, plan or state of behavior, i.e. will happen in an uncertain, unpredictable, indeterminate, accidental, hazardous, causeless, incidental, possible, risky, venturesome, contingent, adventitious, or aleatory way.

In Corpus Aristotelicum (further: CA): lot, fate → *η τύχη*, and random → *τυχαιος*, *συμπτοματιχός* or *συμβεβηκός*. A favorable random event was indicated by *εύτυχιας*, *γούϱιος*, and an unfavorable one by *μοιϱαός* or *ϱιζικός*. In Latin, "alĕa" denotes blind chance, accident, or risk, and this word has survived in French as aletoire. In Russian, we have sloochainyi, in Polish – losowy, in German – zufällig. It seems to the author that words in the classical languages which did not belong to medicine or biology were generally not absorbed into the scientific language of modern times. What is more surprising is the use of the word "stochastic" (in Greek - *στόχαστϱο* means hindsight), both in the sense of random and in the sense of fine tuned to reach a goal – e.g. in the case of modeling, the aim is to obtain the best fit to observational data).

The concept of chance (randomness, hazard, accident, fortuity) now logically means the uncertainty of events and processes or their unpredictability.

In Vol. I of his works, Aristotle referred to random (accidental) causes as being such things that accompany the necessary cause at a given time and in a given situation, but by themselves they cannot cause the effect without the action of the necessary cause.

When analyzing causes, he also mentioned, by the way, that an accidental feature (*συμβηκόs*) expresses neither property, definition, genus, nor aplostic properties. Thus a feature is relative, and uniquely connected with a particular single thing at a given moment of time, and it may or may not be fixed with regard to the subject. In Vol. II, Book II on Physics, there is both a repetition of the conceptualization of accidental causes as features (properties) concomitant, attendant or concurrent to non-accidental causes, and an extension of this idea to non-concomitant accidental features. Accidental, chance events and opportunities are also discussed. They are also referred to as undetermined, unknown godlike, spontaneous, unpredictable, unnatural, rare, automatonic ($\alpha\upsilon\tau\acute{o}\mu\alpha\tau o\varsigma$), incidental, hazardous, coincidental or tycheic ($\tau\grave{\upsilon}\chi\eta\alpha\varsigma$). Hereafter, these qualities will be sometimes abbreviated to AKT ($\alpha\kappa\tau$). Thus, if a chance event occurs, it must be chance that does not disturb necessity. Hence, chance is a tier, a link between an event and the conditions in which this event occurred. A good example of a random cause is, e.g. the following story: Aristotle decided to go to the Agora to buy $\kappa\iota\acute{\alpha}\varrho\alpha$. He bought it, and by chance he met Koriskos who repaid a monetary debt to him. The event of repaying this debt was random, but going to the Agora was a random, though sufficient, concurrent cause of this repayment. So chance remained as the cause of Aristotle's loan being repaid.

It is important to stress that the views of Aristotle on randomness were probably the product of vivid discussion in ancient Greece. The most striking and consistent was the opinion that chance or accident can only be attached to beings who are able to act and express emotions of happiness, pleasure or pain. Hence, small children cannot carry out directed random actions or deeds, since they do not express conscious intentions in their deeds. However, their actions are spontaneously random, due to the possibility of unconscious choice.

The most decisive conclusion and guide for modern modeling pronounced by Aristotle is that no random cause can be an earlier form of a significant (necessary, main) cause. Also, there exist things whose motion is spontaneous, automatic, due to the motion of other things without the possibility of making a choice or being something else. Such motion or action causes reactions against the mover or bearer. This statement sounds remarkably similar to "actions cause reactions against the mover or bearer", which is the basis of Newton's postulate III formulated 1900 years later.

For contemporary modelers, it is worthwhile to recall, that the following idea: "the change of change may only be random", was probably shared by others, apart from Aristotle. Whether this means that models based on $\triangle^2 y\left(t\right) = g\left(a_{(t)}\right)$, where $a\left(t\right), y\left(t\right)$ are real functions, are always stochastic remains to be tested. Another problem to be solved is whether under $\triangle^2 Y\left(t\right) = g\left(A\left(t\right)\right)$, where Y is a formal random object, the expression $\triangle^2 Y\left(t\right)$ can become non-random given a random cause A. Among the fundamental statements regarding causality in Volume 2 mentioned above, we can also find the idea that nothing which is the effect of a random cause can be imperishable, unoriginated, since having a random origin is incompatible with things that always exist. This volume also contains the three following fundamental statements:

bs1) (Anaxagoras): Things only combine and disjoin, nothing vanishes and nothing new appears (repeated, e.g. by Lavoasier);

bs2) nothing impossible exists or comes into being,

bs3) nothing impossible occurs.

It is necessary to find how **bs1**–**bs3** are related to concepts of randomness and chaos, as well as to the non-regularity and spontaneity arising in living beings due to elements combining and entropy increasing as a result of nature being non-durable. By saying that something is random, indicates that something may or may not exist. Therefore, it indicates that something is possible, and as such, is in opposition to the property of something being necessary. For modern modelers, it is particularly interesting to first answer whether necessary causes, facts, events or processes can exist without their concurrent random counterparts, and, in particular, secondly, whether the effects of causes are always random and explainable only by random causes.

If the first case holds, then it suffices to have deterministic models. In the second case, the modern tendency of favoring stochastic models is legitimate. Aristotle preferred the view that each body or being has its *ούσια* part with essential features and *ποιότης* part with changeable (*συμβεβηκός*) properties that are accidental. Heraclitus in turn, stated that *τύχη* governs necessity, which means that he was pessimistic regarding the predictability of future events.

Even the gods were treated by Plato, Socrates and Pythagoras as accidental causes of the activities of living beings. Earlier, however, Anaxagoras warned that an accidental thing, by its essence, does not mix with any other accidental thing – so, in the current modeling jargon, when you formulate a model – do not mix two random factors in one equation if you wish to discern the influence of the individual factors. What enables us to reject his warning remains to be answered. On ontological grounds, while discussing the reasons for the potential existence of elements in the Universe, Aristotle refers to the concept of the possibility that "all that is, may not have been, so *τύχη* exists, since even that which does not yet exist is able to come to existence". In the case of metaphysical arguments, he also distinguished between random sensual feelings about revealed facts and phenomena as uncertain sources of knowledge. As causes of actions, accidental causes are considered to be improper, insignificant, concurrent, auxiliary information regarding the particularity of an effect.

Therefore from the ontological point of view, accidens is understood in Vol. II to be such a thing (being, object, person, fact, event, situation or process) that truly exists, but non-permanently and not-necessarily, i.e. it exists by possibility, but not by necessity. From the epistemological point of view, in turn, accidens means a non-essentially specific feature, attribute that a thing, object (etc.) possesses. Such a property exists due to other features, things, events etc. According to Aristotle, accidental beings or things are caused by the possibility of the occurrence of something transient, non-necessary in a particular state or situation, or the possibility of the existence of unconscious feelings, actions and mental changes. Random beings and things are not the subject of science in the view of both Plato and Aristotle. According to them, the sophists Protagoras, Gorgias etc. studied random features of language, not the essential features of reality.

Some other authors mentioned in CA treated random relations as momentary, weak relations acting without general rules, as relations not fulfilling the necessity rule. However, in his studies of biology, Aristotle clearly saw that the random matching of animals gave random new organisms within a given subspecies or species. He assumed that random causes of random effects cannot occur earlier than the essential necessary causes. Both causes and effects may or may not be autonomously existing accidents, i.e. random beings. It would be worthwhile to check empirically whether these two classical postulates are still valid in research activities.

It is interesting that in Vol. 3 of CA (Corpus Aristotelicum), souls are said to have causes lying outside the random motions of bodies, i.e. that souls are exogenous to such motion. To detect logically random features, Aristotle proposed to express cause, not by using a intermediary, but by a term concerning a specific feature. He stated that the particular senses of an individual perceive the appropriate sensations of the other senses only accidentally (randomly). Due to the linkage between two or more sensual feelings and the random sensual features of a subject of the senses, an error in perception may arise. With regard to optical phenomena, such errors are often met and called optical illusions. Randomness also arises due to the impermanent, non-uniform coincidence (confluence, concurrence, juncture) of unintentioned causes linked to unpredictable situations or aims.

This does not exclude the possibility of random junctures, natural errors, deviations from one's purposes due to the passive resistance of matter. Random features are based on the motion of substance and matter. The singularity of a real object lies in its randomness. Moreover, the random movements in a soul are the source of sporadic facts regarding nature that are random-like. There also exists random coincidence regarding real life events conforming to random dreams or other natural events and signs. Their randomness is revealed by the small chance of their happening in reality.

However, only a few dreams are signs of real future events. Usually, they are only accidental coincidences. On the other hand, their randomness is of another nature than, e.g. the characteristics of heat or body temperature. In biological descriptions, a causality principle is applied which states that the same causes imply the same effects, and that causes which do not fulfill this principle are random causes. In his work in physics, Democritus rejects the deterministic principle of nature, and introduces a random impulse (cause) leading to motion in nature.

Some margin for randomness in the laws of nature is accepted by the Aristotelian school, under the assumption of the indefiniteness of matter and the circle of births and deaths. Random changes in natural bodies are evoked by unknown mechanistic causes and not by purpose-led causes. Thus randomness excites and "lives" in Nature. It is interesting that classical philosophers saw methodological errors in accepting similar exogenous, random effects as results of the same causes. Non-Aristotelian views also treat random events as non-necessary, not *per se*, but still non-substantial. In accordance with the concept of accidens – features cannot act or suffer in themselves, but must act and suffer in accordance with the subject or object to which they indissolubly belong. For modern modelers, we can therefore formulate a criterion for the choice of random features according to which they have to be strongly related to their random effects and weakly connected with the non-random necessary causes of the effects of random and non-random causes.

In Vol. 5 of CA, devoted to ethics, Aristotle considers fate as a random cause of people's actual happiness, fortunes, choice of life goals, tools for realizing these goals, people's actions and important life events. When discussing erroneous actions, he considers unexpected and unintended actions taken due to ignorance and the random causes of unhappy accidents as reasons for such errors. This volume gives a clear warning to psychologists and lawyers by stating that random human deeds are not a good basis for drawing conclusions about people's characters and personalities. Luck is often attributed there to "God loves him", individuals, or unknown laws and abilities, and random events are attributed to non-natural, other-worldly, concomitant or side causes.

In the section of CA devoted to economics, when listing the duties and virtues of a wife, it is said that the unlucky fate of a husband is shared only by the best wives. For the needs of dialectics, Aristotle developed dialectic syllogism, which uses only reasonable assumptions and general *toposes* as a basis for drawing reasonable conclusions concerning random cases in the fields of law, economics and politics, which were so common in the practice of rhetoric.

In the considerations regarding rhetoric given in Vol. 6 devoted to probable cases, things and signs, Aristotle uses the concept of a probable event to mean one that usually happens, and probability is the generally accepted *προτάσις ἔνδοξος* premise that something will or will not happen in most cases. Luck and a strong body give joy in old age. The favor of fate may give good health and/or an accumulation of important goods that excite envy and exceed rational expectations (e.g. coming out unscathed from an accident, rescue from an earthquake).

However, when rhetoricians assess human deeds, they have a propensity to amplify the true achievements of individuals and downplay gifts given by fate. Deeds that are independent of one's own will are done under the influence of fate, compulsion or freely while being under the influence of personal habits or propensities (*τὰ ἤνη*). The misfortunes of fate experienced by an individual can also be used in rhetoric to evoke the listeners' mercy.

In rhetoric, the term randomness is used in connection with *τό δυνατόν*, i.e. the possibility of existence or non-existence, of something acting or not acting. As Agaton stated "to other things fate and need itself forces us". Socrates' opposition to the use of random devices by state clerks does not indicate that in antiquity the view of each men being a slave of money and his fate was not popular. When discussing enthymematic *topos* (called later in Latin "fallacia accidentis"), Aristotle underlined that *topos* consists of the use of a random premiss that "it is the essence of a thing itself which entitles any of its qualitative features". In Rhetorics for Alexander there is a statement that public expenditure depends not only on the people's decisions, but on the favor and disfavor of fate. During debates about wars, when speakers are against them, they should stress that during wars, changes of fate are frequent and impossible to predict. Arguments based on such a concept of probability are contingent with known and familiar emotions and with the habits and abilities of listeners, viewers and readers. To induce belief in listeners, speakers can justify their arguments based on the necessity of a situation or the ordaining of fate. When discussing how tragedies are described, Aristotle stated that tragic events are more striking when, against expectation, they form a clear sequence than when they occur spontaneously by accident. This creates an impression of their intentionality. When presenting his rules for the composition of drama, he proposed using transition from happiness to misery due to *ἁμαρτία*, i.e. a great flaw. Moreover, within a tragedy, the intervention of the gods should be described outside the main contents of the story, i.e. in a prolog or epilog. In Propreptikos, Aristotle introduced randomness as a result of the functioning of the body induced by Reason, and the concept that a good man living according to the rules of Reason is never influenced by fate. It is interesting to highlight the pragmatic consequences of the use of random lots by the archonts, treasurers of the magistrate of Athens (from the times of Solon), as well as the random allocation of Kleystenes from new smaller administration units to larger ones. Later, the appointment of officials and clerks of various ranks by lot was also practiced, not only during the time of Sofocles and Perycles, but also in ancient oligarchical power systems. It was also judicial practice to assign judges to a given judicial case using quite complex schemes of random lots which took into consideration the regions in which they lived and used various physical objects necessary to carry out such a lottery. Sextus (in "Against the Physicists", where he referred to Epicurus using the term accidenticality when he stated that "non-body accidens of first bodies are causes of non-body accidens of compositions". Also, he seconded the Parmenidean view that "non-being does not have accidens" – i.e. stated that a non-being cannot experience anything. Moreover, when discussing definitions of good and evil, Sextus presented the view that no description of the essence of good can be correct, since it is only possible to determine its *συμβεηκός* (accidens).

In Book II of De Civitate Dei Ch. XXIII, Aurelius Augustinus (Saint Augustin) stated that the fates of temporal matters do not depend on the goodwill or reluctance of demons, but on the sentences of the true God. Similary to Aristotle, he believed that accidens are connected with bodies, not the mind, which is part of the soul. In Book V, Ch. I, in turn, the size of the Roman State was not attributed to accidens, fate, some system or unwise order, or a lack of causes, but was a result of God's will. According to him, something is random when it does not have any cause or does not follow from Reason's order. Destiny comes from a necessity that does not result from God's or man's will, or set causes dependent on God's will. Augustin ascribed fate to God's will and called the influence of the star system resulting from God's will "fatum" – so treated such a system as the transmitter of God's will. In addition, he did not share the astrologists' view that stellar constellations and time of birth (even for twins) are the causes of indivuals' fates and life histories. The constellation of stars is fixed at the moment of the conception of twins, so how can twins have different fates? It happens, however, that the beliefs of some astrologists when translated into prognoses may sometimes prove to be accurate – although, according to A. Augustinus, this is a case of the hidden "whispers" of evil ghosts. When disputing with Cicero's stoical views, he shared the view that the prognoses of astrologists are unreasonable, but did not share Cicero's view that the future is unpredictable (i.e. that praescientia and destiny=fatum do not make sense, due to people's will and power being subject to destiny).

Among "causa efficiens", Cicero distinguished between cause based on fatalis (destiny) accident, nature and free will. Augustin accepted the existence of fortuitae (accidental) causes, considering them to be hidden, but dependent on the true God's will (his will also determines natural and free causes, which are also partly determined by angels and people).

Citing the pre-classical words of the Egyptian Hermes Trismegistos, fate may be a precursor of the future events and possible actions affecting people.

In Book XI, Ch V, A. Augustus stated that the creation of the Universe and time did not occur randomly, but due to choice by God's Reason.

When commenting on Plato, he recalled Plato's view that even good and wise souls cannot be eternally connected with their proper bodies due to evil fate.

Discussing Porfirius' views about the founding of the Christian religion and Christ's divinity, A. Augustus stated that Christ's soul was entwined with the souls of others, due to unfavored fate resulting from the fall, and the souls of others were also led by fate not to recognize their faults of faith (see book XIX, Ch. XXIV). In presenting his views on the role of external and material things in life, Augustus stated that one can only have the wealth to satisfy his own needs and also to help others by chance (see. Soliloquia). Here, he also repeated his views on the non-necessity of the joint existence of an accidental feature of an object (e.g. quality) and the object observed.

More focused notes on accidents and accidental features can be found in Avicenna's (Ibn Sina) texts on logics, metaphysics and physics. In his book on logics, he stated that a general feature (or property) may be recognized as being random (or accidental) if it is non-significant, i.e. at least one of the following conditions is not fulfilled: 1) a researcher cannot distinguish between genus and species (general or particular), 2) a researcher understands that the idea of a significant thing precedes the idea of a particular thing, 3) a scientist accepts the assumption that a thing has meaning in itself and not due to a particular concept.

Note 2.1. These three conditions may be summarized as conditions for a lack of knowledge about the subject or object of analysis or a lack of appropriate information. More on this may be found in W. Milo (2013) and the literature cited there.

This also means that a random property is something that is intrinsically linked to a thing, which does not exist in the imagination. Thus random properties are not features that precede the significant features of things, i.e. they result from the significant general features. The evenness of numbers and the natural ability to laugh are examples of such features. Hence, attributes are treated as being random if they are subordinate to essential features, non-significant, non-general and non-species defining features. Random attributes are distinguishing features of specimens belonging to the same species. It should be remembered that a random general attribute (e.g. the ability to laugh) may belong either to one general class or to more than one class (e.g. the ability to move or color). When characterizing states of randomness, Avicenna distinguished between two kinds. The first originates from the substance of a thing and the second from a comparison of one thing with another. There are two subkinds of randomness of the first kind. The first subkind consists of kammiya (quantity) as the cause of existence in substance, in its growth, decline, division and dimensions. The second subkind is characterized by qualitative states (kayfiya), such as health, wisdom, knowledge, whiteness, voice, heat, triangularity and weakness. The second kind of randomness covers 7 subkinds characterized by place, time, position, possession, activity and passivity. They show the structure of random attributes attached to the observed states of a thing. Geometrical concepts of how to construct (in one's imagination) a continuum or plane are treated by Avicenna as random objects with random attributes. The features of numbers are assumed to be random, due to the fact that their ontological counterparts in bodies are also random. For example, the unity of bodies and the unity of numbers assigned to them are random. The same is true for the shapes of bodies and their colors. Without a subject or object, there are no shapes, colors, quantities, magnitudes, i.e. their accidens as random states. Thus the states of these categories are accidental states. Since necessary being is not the substance of being, it does not belong to any of the categories of states mentioned above as being random. Moreover, elements of bodies have random properties due to their nature, form and interaction. It is worthwhile to recall that accidental features, as opposed to significant ones, were often considered by post-Socratic thinkers, when they were discussing the properties of good and evil. Many comments on this topic can be found in the works of Sextus Empiricus where he recalled the views of Xenocrate, Plato, Timon, Pyrrho and Epicure. In his discourse, he stated that their definitions of these concepts do not define the essence of good or evil, but only their $\sigma\upsilon\mu\beta\varepsilon\beta\eta\kappa\acute{o}\varsigma$, and that it is impossible to reconstruct their essence from their accidental properties.

The problem and concepts of accidental beings were also considered by Thomas Aquinas, Giles of Rome and W. Ockham. Avicenna treated the essence of a being in itself as only possible in the form of accidens (e.g. the sex of a person). On the other hand, St. Thomas proposed that accidens, by its nature, is not necessary, yet there exist things that are accidental to essence, but do not change it. Each accidens, according to him, adds to the essence of a thing and something which is accidental is subordinate to that which is essential and everything in God is non-accidental. Although T. Aquinas (see his discussion about faith) affirmed God's supreme power, he did not exclude the occurrence of random events, fate or lower-rank causes if they resulted independently with regard to intention and aim, but within the aims of an upper-rank cause. To give a contemporary example, suppose the sales director of a given company sends two auditors to the same subunit in another city on two consecutive days without announcement and only he knows about it. Hence, the auditors meeting workers in the subunit is random according to both of them and the auditors. However, from the point of view of the director, this event is foreseen, certain. This means that, according to St. Thomas, an event is random from the point of view of an observer, if he or she cannot predict, forecast or foresee this event. For us, however, there are no random elements in the ontic parts of an event from the past. No randomness has occurred whatsoever, it is just the result of a natural process.

As Boecius stated, fate is invariable design in relation to variable things. According to St. Thomas, although God's providence to things is immovable, the effects of the most direct causes are either necessary or accidental, and if one cause is necessary and another is accidental, then the result is accidental. Therefore, the birth and death of living bodies in nature are accidental, since their lower causes are accidental, and some results of God's providence realized by lower causes will also be accidental, despite God's omniscience and the wiseness of God's intentions with respect to future accidental beings. Moreover, the randomness of things does not exclude the immovability of God's providence. Although random causes, in themselves, do not evoke effects, thanks to appropriate assistance, when in support of non-random causes, they bring firm and valid random effects. In modern language, it is sometimes perhaps relevant to assume that random causes are correlated causes, and non-random causes are uncorrelated causes. A source of randomness related to apparent discrepancies between the Old and New Testament can, as St. Thomas stated in "Contra errores Graecorum", result from inappropriate interpretation of parts of the text due to a lack of knowledge in ancient and classical languages and general ignorance. The use of lots and signs of fate were discussed in St. Thomas' book "De sortibus", which was devoted to predicting future things and events of concern, apart from the existence of finite numbers, known stars and God. Lots were usually drawn in personal matters, e.g regarding threats of floods, drought, earthquakes, i.e. regarding unpredictable distortions of nature's laws and order which are impossible to foretell and counteract in favor of the persons affected. Things and goods held in common and partitions were often the subject of drawing lots (as well as important offices of power, such as the election of Saul to become king or of Zacharias to the priesthood) or similar procedure for partitioning the goods among the candidates. Lots, therefore, can be used for partitioning, consultation and divination. In the final case, such lots were used as a form of mediation between God and his prophets. Predictions of the future are also given by astrologists, augures, seers, palmists, divinators, dice (bone) throwers (cleromancers). Thus, with regards to lotteries, the decision to use one is taken when it is outside of a human's abilities to know what will happen to someone, to his future being. Based on Cicero's arguments, it can be stated that lots are useless in divination and augury, since both future things and accidental things are unpredictable. Drawing lots to make a partition, however, makes sense if it is acceptable to all those interested in it. On one hand, this does not eliminate fears about the future, but it does eliminate fears about the present situation. Thomas Aquinas did not share the views of either of Cicero, or, as he said, of stoics pharisees, Pryscillians, or ancient physicists, according to whom human affairs are dependent on the stars. Thomas Aquinas highlighted the active role of the organizers or performers of draws, as well as the propensity (often unconsciously) to believe in the results and evoke future events in accordance with the results of a lottery. Due to these phenomena, astrologists could often "predict" events related to groups of people, but they were not often successful in foretelling single events for specific individuals (e.g. when digging a grave to find treasure).

One can also approach random events (fortuities) in the same way as e.g. the results of someone purposefully setting a trap, snare, pitfall, ruse or trick, i.e. purposeful plans and intentions to fool or trap, or catch naïve, stupid, silly or lazy people or to punish evil ones. So things that seem to be random to the target, were intentional, conceptional, inventive plans to achieve a specific goal by those who planned and set the traps. The question remains open as to whether these goals and the conditions for their realization are random or not. According to St. Thomas, all is governed by God's mind and will. Yet he did not clearly state whether randomness functions, from the fact that he did not criticize Dionisius for saying that God's gifts can be treated as random by their recipients. Hence, he seems to have assumed that such an interpretation is valid. In the book on "De substantiis spearatis", Thomas Aquinas again stated that for a governor who has knowledge about the presence of enemies on a road, sending serviceman to ambush them is certain, but for the serviceman such an attack is random. Similarly, God's providence is stated to be the cause of all intended results. Some necessary results will be evoked by necessary causes, while other results will be random due to the random causes appropriate to them. Commenting on Proklos' views on randomness, Thomas Aquinas in his book "Liber de causis" began by distinguishing between primary and secondary (derivative) causes, by the facts that the first help the second to act and secondary causes are effects of primary causes. The power to act of the first is more influential, far reaching and general. Despite the fact that a secondary cause is an effective cause, it is less general (more particular). Thus secondary causes add specific properties to things. Proklos distinguished causes according to *causa per se*, and *causa contingens* (or *accidentalis*). The action of a *causa per se* brings about the final result through the action of all its accompanying indirect cases. The action of accidental causes, in turn, lead only to direct effects without opposing the first *causa per se*. Thus, the *causa per se* evokes all the indirect causes to reach a final effect. However, accidental causes act in another way, i.e. the effect of the most direct random cause does not spread to the other objects of causation covered by a *causa per se*, but concludes at the object appropriate to it.

Aegidius Romanus (Giles of Rome) shared similar views on accidens. In his original book "Theoremata de esse et essentia", he distinguished between "unum or ens per se, and, unum or ens accidentale", as well as between "unitas essentialis and unitas accidentalis". For example, the features of being human and being musical are in accidental unity, since musicality is connected with concrete individuals. So "ens per accidens" means something is added to the essence of ens. In theorem XIV, Aegidius stated that "each accidental form gives its object some existence, though the existence entailed by such a form cannot be different from the entire object composed of its form and matter or substance (or subject)". He also affirmed that each form, whether it is substantial or accidental, acts in relation to the subject which it forms or modifies. Moreover, unity of essence or nature can never arise between an accidental form and its subject and there is a real difference between substantive and accidental existence. The subject *se ipso* is determined by its accidental form and vice versa. The existence of an accidental form is not self-generating or caused by the substantive form, but by the totality form. Each creature has accidens, because only the first ens is perfect and non-accidental.

It is also interesting to consider the views of W. Ockham on accidental properties, things and causes. In his Summa logicae, in Section I. 17 he answered some questions about the theorem that “if no generality is substance, then all generalities are accidentia, and so all categories will also be accidentia (including substance)”. He distinguishes between accidentia as generalities and tangible signs of accidentia that consist of substance. He relates substance to accidentia, an accidental property which is not separable from the observed thing. In Section I. 25, Ockham stated that accidentia is the fifth type of generality, and it is such a feature that a subject may or may not possess, while not ceasing to be itself. This means that accidentia denotes something really rooted in the substance (as heat is a necessary characteristic of a fire). Such a notion indicates that accidentia lie outside Reason and so exclude its meaning as a fifth generality. Another meaning of accidentia is all that can be or cannot be predicated about some existing subject or object while preserving the truth of a statement and relative *modus essendi*. A third way of understanding the term accidentia is as predicates (in a logical sense, Ockham uses the term “predicamentum”) about something which might be – due to changes in the subject – discussed, or something else – asserted or denied. Thus, according to Anselm from Aosta, numerous relations are accidentia, i.e. they may arise and disappear. The fourth meaning of accidentia as predicates states that something which does not denote an absolute thing that is a part of subject but a relative thing caused by a local change in discussed object or local motion in parts of the thing is accidentia. Thus these four interpretations by Ockham of accidentia are in reference either to ontic things (the first two) or the grammatical parts of sentences (the last two). Despite this, Ockham distinguished between accidentia which are separabie (by natural forces) from an object without destroying it and inseparable. The first denote separable features of subjects, and the second inseparable ones. Commenting on Aristotle’s distinction between *ens per se* and *ens per accidens*, Ockham, rightly, stated that it is connected more with the form of the predicate than other matters. The first is thus a necessary (significant, certain, primary, general) predicate about an object, and the second is non-necessary, non-significant, non-certain, secondary, particular predicate.

Since the author of this text did not have access to the original texts of, among others, Peter the Spaniard (1205–1277) and R. Grosseteste (1168–1253), their contributions will be omitted. It is, however, possible that W. Ockham had access to them, although, similar to many other of his predecessors and later writers, he did not refer to all the original authors of concepts and views. The same can be said about John Duns (commonly referred to as Duns Scotus), who wrote, e.g. the treatise "De primo Principio". In part 4.10, he affirmed that pure perfectness of the highest nature (God) is not accidental as a property, but proper and necessary. In part 4.15, in turn, he stated that the only source of *accidentia* actions by man is his free will or what accompanies this will. Here accidentia is not all that is unnecessary and transcient, but only that whose negation may come into existence, when appropriate. On the other hand, the primary origin of accidentia could be the actions of the primary cause (which due to man's ignorance is unknown). This is more clearly stated in Theorem Five (T 4.23), i.e. "whatever the primary cause generates by acting as a direct cause, it does so in an accidental way".

Science advanced rapidly in the XVII century. From the point of view of studies on accidentia, it is worthwhile to describe what was written by Rene Descartes, B.B. de Spinoza, G. W. Leibniz, A. Arnaud & P. Nicole. In his "Meditations de prima philosophia", Descartes, like many of his predecessors, considers only substantial bodies as a source of accidentia. This also means that, according to him, mental constructs regarding abstract objects or religious concepts are clearer than ideas concerning material, physical elements, features or facts, since they relate to accidentia.

When answering a group of objections regarding the search for truth, Descartes, in his seventh postulate, stated that when seeking the truth one should remember that the truth is found non-accidentally if clear reasoning is applied. However, by applying unclear reasoning, due to errors by the senses, unclear hypotheses, superstitions or ignorance – the truth may be found only accidentally. Axioms VI, IX, X, which directly follow these postulates, also refer to accidentia. According to Descartes, substance has more reality than accidentia or modus, and infinite substance is more real than finite substance. This leads him to state that there is more objective reality reflected in the concept of substance than in the concept of accidentia. Moreover, it is more important to create and preserve substance than to preserve the attributes of substance, especially accidental ones. According to Descartes, the finiteness of physical things implies the acccidentia of such things. In his texts on astronomy, Descartes did not reject real accidentia and he noted that they are usually of an unknown nature.

He shared the view that real accidentia do not conform to the views shared by some theologians and philosophers, and that reality is assumed in order to explain the feelings of our senses synthesized by the activity of the brain and next transferred into mental activity.

By accidentia, Spinoza understood the divine guidance of human affairs with the help of unexpected, unknown causes (see §11 of Ch. III of Tractatus theologico politicos (1670)). He uses the concept of gifts of fate to refer to means used to secure the livelihood of individuals. These means are exogenous things dependent on the forces governing exogenous causes and especially wise leaders of a society. As an example of accidental causes, Spinoza cited Kohelet in relation to Solomon's concept that all is governed by accidentia. According to Spinoza, prophets had a vague idea of how the order in nature and human fates are related to God's providence. In Ch. X, §22, it was noted that "Queen Ester (Esther) ratified all which concerned the festival of Purym (fates)". Ch. XVII, §45 is much more interesting to modern practitioners of statistics and econometrics, where it was stated that Moses' destiny was to form an army (from 12 tribes) in order to conquer the state of Canaan, to divide this state into 12 parts and, by random sampling, assign them to the 12 tribes. Under the common laws of nature without any religious interpretation, as king Solomon says (see Ch. XIX, §8), the same fate awaits believers and unbelievers, sinners and non-sinners, and there is no place for justice or mercy.

In Tractatus politicus Ch. X, §1, Spinoza presented a view adapted from Machiavelli, that in any monarchical or aristocratic system there will be times of dangerous degradation within the system that need planned or random actions in the form of either wise legal reforms or the activity of great statesmen, in order to avoid the fate of the state collapsing. Although Spinoza, similarly to Descartes, adhered to deductive inference, he was conscious that (see Tractatus de intellectus §19) fixed, variable, accidental experience exists that is useful in any cognitive process. There are comments on inferring a cause from an effect (earlier than the statistical solution presented by Bayes' theorem), though they are unclear, ambiguous and very tentative. The only content derived here about inferring a cause from its effect is the notional property of a cause, but not the material content of the causal object. Spinoza also considers, in §84, 91 of this treatise, a distinction between a true idea, an error, and a fictitious idea that originates in a variable phantasia understood as a source of random feelings incited by exogenous random causes.

Readers favouring a more formal style of writing can read the text "Ethics". Theorem 15, Part III states that "Each thing may be, *per acci-*

dens, the cause of joy, sadness or desire", which means that Spinoza assumes that a secondary cause, accidental cause, possesses some usefulness and validity. Moreover, in Theorem 50, he states that "Each thing may be, *per accidens*, the cause of hope or worry".

Discussing the distinction between accidental and possible things in Part IV of his Ethica, he defined them as follows:

"I call particular things accidental, in so far as, considering their content, we cannot find anything that is necessary for their existence or non-existence. I call such things possible, in so far as, considering their necessary causes of origin, we do not know whether they were designed strictly in accordance with this origin". He also considered opposite emotions that are opposite by accidens, and are less intensive than necessary and possible past emotions, (see Th. 11–13). An interesting note in the form of Th. 17 concerns the statement that "Desire that originates from the knowledge of good and evil, in so far as it concerns accidental things, may be even more easily restrained by things present now".

An analytical, critical discussion of Descartes' philosophy and scientific method was carried out by A. Arnauld (1612–1694), who – together with P. Nicole in the book "La logique, ou l'art de penser (1662)" proposed an extension of the a priori-deductive method of Descartes. In Chapter I of this book there are comments on common accidental properties expressed by a secondary predicate, which is subordinate to the primary necessary predicate connected with the concept of substance. So saying that "Aristotle was a biologist" uses the connotative term "biologist", while being an accidental property creating a fifth generality not belonging to the essence of Aristotle.

More detailed comments than in Arnauld-Nicole's book are contained in G.W. Leibniz's (1646–1716) texts. In "Nouveaux essais sur l'entendement humain", he defined secondary qualities of given human abilities or physical strength which cause certain feelings of the senses and considered perceptions of the idea of such feelings. He stated that accidental things or events are not necessary, though they are determined. Accidental truths, in turn, are not necessary at all, and even their unions do not always have the status of absolute necessity, due to the way of distinguishing between necessary consequences and accidental consequences. Hence, geometrical and metaphysical consequences are enforcing, but moral and physical ones are only inciting or inclining. He proposed the future development of a calculus for probability and truthness based on appropriate logic. Such calculus would be useful, especially in the case of the analyses of accidental species to which particular *individua* may stop belonging after certain changes caused by external forces. Analyzing various degrees of our knowledge, Leibniz stated that the truth of reason is necessary, but truths about facts are accidental. He acknowledges intuition and proof as important sources of knowledge, but, according to him the rest are only endogenous beliefs and opinions.

The most advanced of Leibniz's approaches to accidens, close in meaning to randomness, is seen in book IV, Chapter XVI, which is devoted to degrees of conviction. Here, the concept of random covers dice games and the fair value of a game after its interruption. He commented on the letters and written discourses of de Méré, Pascal, Huygens on the calculation of probabilities, as well as on their usefulness in solving life insurance problems, as J. de Witt showed in 1671. According to Leibniz, the fate of each man lies in the substance of his embryo, which includes a record of his future. In his letter VI to Arnauld, he placed liberty and accidentia outside of the influence of God as a primary cause in his notion of the universe, and rightly assumed the usefulness of the concept of accidentia. He stated that "absolute rejection of pure possibility destroys the rationality of accidentia". According to him, the reasons for accidental truths incite, but do not enforce decisions (acceptance, verifiability, decidability). Referring to examples of *unum per accidens*, he stated that the reason for such unity is an exogenous force accidentally enforcing this unity (see: letter IX).

Continuing our descriptive and philosophical description of historical understandings of accidentia, it is also worthwhile to present I. Kant's (1781) views on this concept given in the book "Kritik der Reinen Vernunft". In subchapter II of the introduction, he posed the following question: "from where does experience take its certitude if all its rules are empirical, therefore, accidental?" This means that, for Kant, empiricism and accidentia are intimately related. Much more is said in Part II, Chapter III.4 of book II devoted to presenting all the synthetic principles of intellect. Analyzing the principles of a knowledge system, he stated that it is impossible to prove the truth of a synthetic statement of the form "all that exists in an accidental way has some cause", although this sounds correct. In this case, the notion of an accidental thing contains no modal category apart from the category of relation to something that can exist only as the result of something else, due to the assumption of the existence of changes in them. Hence, the fact that a body has come to rest following some motion does not prove the accidentality of its motion simply on the logical ground that motion is opposite to rest. To prove accidentality, it is necessary to justify that the body could have been earlier at rest instead of being in motion. Thus, if one considers a thing to be accidental, and states this, then this is equivalent to saying that the thing has a cause. In Part II, Kant also defined a cause as a condition for something to take place or happen. He called something accidental if it is conditioned by being connected to something general. Anything which is not conditioned in such a way is called necessary. Kant's notes concerning the fourth antinomy concerning pure Reason are much broader and more consistent. He placed accidentia in two contexts. The first comes under the hypothesis: a necessary being exists, independently of whether this being means the world itself or something else. However, the decision as to what it is needs to distinguish between the notion of accidental beings possessing an intellect and notional connections between them and this necessary being which preserve the empirical laws of causality with the highest member of the chain being the necessary being. However, according to Kant, the conclusion that changes evoked by empirically existing causes in the world justify the existence of empirical accidentality is not consistent with the above methodological procedure. Therefore, it was accepted as being valid to assume that accidental is anything for which contradictory opposition is possible. Moreover, from empirical accidentality one cannot draw conclusions about notional pure accidentality, i.e. oscillation between opposite notions does not prove accidentality according to the notions of pure intellect. Such change only proves that a new state cannot come into existence by itself without a cause existing at an earlier time. A cause, even if it is necessary absolutely, must exist in this way in time and apply to many phenomena. The notes concerning the antithesis, i.e. that "such a necessary being does not exist" are aimed at accepting a cosmological argument derived from the accidentiality of the state of the world and changes in it that prove that a primary cause does not exist, since the past contains all the accidentally conditioned restrictions that are determined within the processes describing the world and which determine the occurrence of spontaneous events. The spontaneity of events autonomically incited by someone defines real accidentiality and, most often, it consists of the accidental being of the substance itself being derived from the necessary substance.

Kant also described conditions under which it is reasonable to distinguish between the accidentality of form and the accidentality of matter and whether it is wise to connect them with the notions of the natural order, goals and consistency. It must be said that, in the case of physically existing objects, conditioned empirical facts are always treated as accidental, so their condition can only be recognized as relatively necessary. The basic principle of studying nature is to reason from empirically accidental results to a possibly accidental cause.

Kant also included God as a first cause, which helps to explain all that behaves accidentally. He rejected the usefulness of the concept of accidentia and their signs in proving transcendental statements. Kant also considered the usefulness in transcendental thinking of such notions as accidental goals, accidental (pragmatic) beliefs and faith at different levels.

The views of Kant regarding accidentia were published in his “Kritik der Urteilskraft”.

An interesting, although now doubtful point, is that he gives respect to the a priori conditions of the human ability to reflect, as well as the accidental consistency of a subject with the powers of human cognition. He also states that the objective purposefulness of nature serves as an argument for the accidentality of the purpose or form of a given thing observed due to the existence of human Reason. Epicurus’ view that blind fate is sufficient reason to explain this consistency is unacceptable to Kant. He also criticizes Spinoza’s use of inherent accidens as belonging to the source of being.

In part II, Chapter 2 of his treatise, Kant acknowledged that a deed which is morally absolutely necessary physically is considered to be totally accidental, i.e. something that is necessary should take place, but for some reason often does not take place, due to the fact that Reason expresses this necessity not by existence or an event, but by something that should be. Also, Reason, by its qualities, treats particularities as consisting of something accidental in reference to generalities, and demands – according to a union of particular natural laws – an accidentality rule, which he called purposefulness. Moreover, the derivation of particular laws from general ones is, from the accidentality of particular laws, impossible. In Appendix §§80–81 he affirmed the existence of accidentality in everything that is thought out as a possible goal placed in a causal relation. Analyzing the rules of reasoning (see the Introduction to the book “Logik”), Kant discussed such necessary and accidental rules. The second kind of rule depends on fixed subjects of recognition, the accidentality of thinking about such subjects and the choice of rules and subjects. However, logical rules need to be fixed according to the necessary use of intellect.

He also distinguished between necessary (significant) and accidental (non-significant) features. The first always have to accompany the thing being presented and analyzed. The second kind of features are those which can be separated from the essence of a notion or thing.

2.3. Final remarks

It should be remembered that Kant’s considerations were probably more inspired by D. Hume’s book (1748) than by the concepts discussed. Hume repeated many times that experience, empirical facts and established practice in making observations and the reasoning based on them have increased our knowledge about Nature. However, a lack of knowledge is sufficient reason to accept the usefulness of the concepts of randomness and probability in scientific language. In an analysis of all these ideas, he strongly recommends using the relations of similarity, contiguity and causality. In order to analyze facts, he prefers to use induction and assume that the relations between facts, events and phenomena may be arbitrary and accidental, both in representing succession in time and space, as well as contemporaneous coupling.

It is easy to see that our text does not present all of the views on randomness presented before the beginning of the XIX c. or anything later. We have excluded approaches with foundations in mathematics, which are especially prominent in probability theory. Such approaches have been discussed in many articles and books (see, e.g. I. Hacking (2006)).

In the XIX, XX and XXI c., there have been many popular and prominent trends in philosophy, the natural sciences, and, in particular, in the social sciences that have promoted an ideology of indeterminism and the relativism of truth. In the author's view, the concepts of random cases, events or processes are simply expressions of our ignorance regarding the real cases, events and processes considered, i.e. a lack of knowledge of their causes. If we knew them, then these concepts would not be useful in any descriptions, explanations or predictions. This does not mean that the calculus of probability, statistics, econometrics or biometrics is not useful. They are very helpful in finding estimates of means, variances, covariances and quantiles of probability distributions which deepen the precision of our descriptions of empirical historical processes, facts and events in many fields of life and science, especially in descriptions and explanations of economic and financial market data. Methods and models employed in econometrics and economic statistics are especially useful in these research activities. In this book, such concepts of randomness will be applied in chapters 4–10.

Bibliography

Aristotle (Arystoteles) (in Polish), (2003), Vol. 1–7, Wydawnictwo Naukowe PWN Warszawa.

Arnaud, N. P. (1662), *La logique*, Paris.

Aurelius Augustinus (354–430), *Soliloquia*, De Civitate Dei.

Avicenna, Ibn Sina. (1974). *Logics, Metaphysics, Physics.* PWN.

Descartes, R. (1897–1910). *Oeuvres de Descartes.* T. VII, Paris.

Duns, J. [Duns Scotus] (1950). *De Primo Principio.* Theoremata (in Opera Vol. I – Vaticana).

Giles of Rome (1952). *Theorems on existence and essence* (trans. M. Murray Milwaukee).

Hacking, I. (2006). *The emergence of probability.* Cambridge University Press, Cambridge.

Heller, M.(2011). *Filozofia przypadku.* Copernicus Center Press, Kraków.

Kant, I. (1930). *Kritik der Reinen Vernunft.* Meiner Leipzig.

Leibniz, G. W. (1765). *Oeuvres philosophiques.*

Maryniarczyk, A. (2000). *Akcydens.* Powszechna Encyklopedia Filozofii, PTTA – Lublin.

Milo, W. (2013). Losowość a chaotyczność. *Przegląd Statystyczny*, s. 425–445.

Ockham, W. (ca. 1323). *Summa logicae.*

Poincare, H. (1908). *Science et Method.* Flammearation, Paris.

Sextus Empiricus (1998). *Adversus mathematicos.* OUP, Oxford.

Smoluchowski, M. (1923). Uwagi o pojęciu przypadku w zjawiskach fizycznych. *Wiadomości Matematyczne*, Z. 27, 27–52.

Spinoza, B. (1670, 1677). *Tractatus theologico-politicus, Ethica.*

Suarez, F. (1597). *Disputationes Metaphysicae.*

Thomas Aquinas (1964–76). *Summa theologiae.* London, N.Y.

Chapter 3

Isomorphisms, homeomorphisms, models and modeling

Summary

This chapter reviews the theoretical and practical arguments concerning the roles of isomorphisms and homeomorphisms as fundamental categories which are useful in theory regarding models and modeling. These categories are logical, gnoseological, algebraic and topological[1]. Acquiring a deep knowledge of them leads to be able to understand the abstract analysis of theoretical and empirical models used in the natural sciences and the humanities/social sciences. Since modeling is now very popular and recognized as a valid method of scientific study, we try to formulate principles for the modeling of actual or virtual events, phenomena and systems.

In this chapter, we are interested in ways of recognizing structural regularities in the functioning of systems. The content of this chapter is closely related to the discussions presented in Chapters 1, 2, 4–9.

Keywords: Isomorphism, homeomemorphism, models, modeling, principles

[1] See, e.g. A. Grzegorczyk (1969), N. Bourbaki (1964), R. Switzer (1975), J. Gastew (2009).

3.1. Introduction

Categories of isomorphisms and homeomorphisms can be defined and used in formal ways by mathematicians and logicians or by using everyday language. G. Frey (1969) presents a different approach. These categories are related to the category of model, which is now used in almost every scientific field.

It is possible to define a model by constructing a relation of "being a model" as an equivalence relation, i.e. a relation which has the properties of reflexivity, symmetry, transitivity and heritage. Thus by a model M of an object O (or original object OO), we understand another similar object, i.e. possibly an ideal copy (an ideal model of the original object). Do a model and the original object need to be necessary equally justifiable bases for inference? In general, no. However, it is a fact that a model can be simpler than the original object with respect to some features, which makes models so attractive in academic studies (especially in simulations). This means that when modeling we may depart from the property of symmetry, i.e. that a model M is similar, with respect to chosen features, to an object O, or CO (Copy Object) to signify chosen features of O, and vice versa. However, it is also possible that an object O is simpler than a model M with respect to chosen features – so symmetry may be preserved in some respects. The intuitive meaning of an isomorphism as a relation also takes into account the properties of reflexivity, symmetry and transitivity and is close to the intuitive meaning of a model. Such an interpretation narrows the gap between the notion of a model, isomorphism and conditions for the adequacy of a model or how it reflects real objects.

The intuition behind homeomorphism as a relation takes the properties of reflexivity and transitivity as properties of primary importance.

Among others, G. Frey (1969) and W. R. Ashby (1958) propagated this intuition to fields outside mathematics. One generalization of the interpretation of a homeomorphism representing a model was to relax the condition of the symmetry of similarity between M and O. However, the similarity of M to O may refer to one, more or all of the possible characteristics of the analyzed object O. Hence, similarity may be interpreted as being very narrow, partial or universal.

It is necessary to relate the concepts of identity and identization to isomorphism ($Isomm$), homeomorphism ($Homm$), model (M) and object (O).

In everyday language, instead of "identity", we use such concepts as: "equivalence, equality, sameness, consistency, indistinctiveness".

Twins, electrons, coins or banknotes of equal value, books (with the same title, author and publisher), are groups of identical or similar objects in an appropriately defined group. With respect to the size of physical objects, there always exists a distance $d_{ob_{12}}(O_1,\ O_2)$ between an observer ob_{12} and objects $O_1,\ O_2$ for which the objects appear identical in size (standardizing the distance to the first object to be equal to 1), i.e. $O_1 id_{ob_{12}}\ O_2$. For a distance $d \neq d_{ob_{12}}$ the objects will be dissimilar in size, i.e. non-identical, different. Assume that by the use of *principium identitatis indiscernibilium* (Leibnitz) or the principle of individuation (confirming the unicity and non-repeatedness of the individuality of each abstract or actual object) we can identify ontologically different objects. Hence, it is possible not only to classify objects, but also to study the essence of their differences more deeply, and to establish intervals for the values of an object's characteristics that confirm the justifiable closeness of a model to the object (system, event, phenomenon, etc) studied.

In Section 2, an intuitive and a more formal characterization of $Isomm$ (isomorphisms) and $Homm$ (homeomorphisms) will be presented.

Section 3 will discuss the tacit correspondence between homeomorphisms, isomorphisms and models.

In Section 4, a discussion of problems and principles related to modeling will be carried out.

3.2. Homeomorphisms and Isomorphisms

3.2.1. Intuitive characterizations of Homm & Isomm

Two approaches are applied in scientific analysis: from the general to the particular, or from the particular to the general. Since our analysis in this section is abstract, it is more convenient to choose the first approach. This means that we start our discussion from presenting the category of homeomorphism – hereafter abbreviated to "$Homm$" $\equiv \{homm_j\}_{j\in J}$, as a relation between two objects (domains, fields) similar in shape, where one object is chosen to be the original object (OO) and the other is its copy (CO), i.e. is similar, to an appropriate or satisfactory degree, to the original object.

Intuitively, it is possible to distinguish the following meanings of $Homm$.

1a) $Homm\,(OO,\ CO)$;

1b) $Homm\,(CO,\ OO)$

2a) $Homm\,(CO_1,\ CO_2)\,|\exists\ Homm\,(CO_1,\ OO)\,, Homm\,(CO_2,\ OO)$

2b) $Homm\,(CO_1,\ CO_2)\,|\exists\ Homm\,(OO,\ CO_1)\,, Homm\,(OO,\ CO_2)$

3a) $Homm\,(F; CO_1,\ CO_2)\,|\exists\ Homm\,(F_1; CO_1,\ OO)\,,$
$Homm(F_2; CO_2,\ OO)$, where $F \equiv F_1 \bigcap F_2$ is a set of functions (mappings, correspondences, operators), with the appropriate properties i.e.
$f_i:\ CO_1 \underset{into}{\longrightarrow} CO_2$ or
$f_i:\ CO_1 \underset{onto}{\longrightarrow} CO_2$, $F \equiv \{f_i, i \in N\}$

4a) $Homm\,(F;\ R;\ CO_1,\ CO_2)\,|\exists\ Homm(F_1;\ R_1;\ CO_1,\ OO)$,
$Homm(F_2;\ R_2;\ CO_2,\ OO)$.

In the above propositions $Homm$ represents a set of specimen, more precisely, single form homeomorphisms $\{homm_j\}_{j\in J} \equiv Homm$, and R is a set of non-functional relations (for e.g. equivalence, reflexivity, symmetry, transitivity, connectivity, openness and countability).

In mathematics, one can choose as OO e.g. a given space (topological, metric, covering, projective, connective, Banach, Fréchet, Hausdorff, Hilbert, etc.), ring, group or algebra, and as CO, CO_1 and CO_2 one can choose other spaces, rings, groups or algebras.

In the natural and social sciences, as well as mathematical economics, by OO we often understand actual bodies or systems (e.g. the Universe, Earth, a gas, animals, oceans, air, a human, heart, human brain, human body, population of bacteria, virus population, predator and prey population, ecosystems, economies, markets for goods, financial markets, raw-material markets). By CO_1 and CO_2 we understand either appropriately defined sets, spaces, rings, modules, groups, lattices, algebras or their subsets, subspaces etc. It is also possible to consider functions, non-functional relations and mathematical descriptions of their domains and codomains as elements of CO_1 and CO_2, i.e. elements of CO_1 and CO_2 will be models of the original object OO. Furthermore, maps describing these functions and relations will be treated as meta-models designed to simplify the classification, interpretation, analysis and formal generalization of models.

Homeomorphism is something more than identity, great similarity, or sameness in the construction or structure of an object. It must be remembered that strict identity (i.e. isomorphism) of shape assumes that the structures of CO_1 and CO_2 are isonumerous (in terms of members, elements), i.e. the cardinalities of CO_1 and CO_2 are equal, i.e. $\nu\,(CO_1) = \nu\,(CO_2)$ in the case of finite sets, where ν denotes the number of items in a set or $card\,(CO_1) = card\,(CO_2)$ in the case of infinite sets. In addition, the elements must be the same in terms of their characteristics, i.e. form, volume, color, and shape (from all perspectives). When we fully reject any considerations of the nature or essence of the elements of a set, as is done in set theory, sets only differ according to their cardinality.

In a mathematical description of objects in the real world, the preferred use of $Homm$ is as in (3a) or (4a), that is to say that by a concrete $homm_j \in Homm$, we shall understand a map (function, correspondence, mapping, operator, functional or operation), i.e. $homm_j \equiv f_j, \quad \overrightarrow{f}_j$.

In the case of $Isomm \equiv \{isomm_{i \in I}\}$, by a concrete $isomm_i \equiv \varphi_i$ isomorphism, we shall understand a more restricted, in properties, function (or: map, correspondence, operator, operation).

If we possess, not only quantitative information about CO_1 and CO_2, but additional, strict qualitative information, then we may also consider both $Homm$ and $Isomm$ with respect to specified qualitative attributes and predicates. In general, different $Isomm$-descriptions with respect to different qualities do not correspond to each other. It can happen, however, that a single description is useful to an observer of real systems in the case of two different research situations, especially when this observer sees OO as an $Isomm$-impression. In many research situations, using the less strict $Homm$-approach is more conservative than the more strict $Isomm$-approach to modeling OO. For example, using a $Homm$-approach may be treated as obtaining a similar view to using a $Isomm$-approach, but from "a moving ballon, airship, train, car or airplane window" or "at dusk or in fog".

When we use $Homm$ as a research tool for studying either real world objects or their mathematical models, we may treat the results of its use as either an illustration of the links between originals and their copies, or describe the links between one family of models and another family of models. Hence, $Homm$ may by interpreted as a display of homeographic pictures or photographs of the links between OO and CO – whatever interpretation we give to the elements of OO and CO. Such a display may or may not be in color. In the case of color displays, we should distinguish between "white-weak yellow", "dark bronze-black", "blue-aquamarine" and "black-white" research situations. The greatest contrast, and thus easiest to identify, is the "black-white" situation. Such actual or metaphoric situations illustrate the subtle effects of using $Homm$. From what has been said, it can be seen that $Homm$ is likely to be used by realists and $Isomm$ is likely to be more often used by idealists.

However, both types of research must pay attention to the structure of the objects studied. We recall that the $Homm$-concept covers both the cardinality of OO, CO, CO_1, CO_2, F, and R, as well as the shape of their "architecture".

When building a model M, we should remember the following rules:

r1) $\nu\,(CO) < \nu\,(OO)$,

r2) $\nu\,(CO) < \nu\,(classes\ OO)$,

r3) $\nu\,(classes\ CO) < \nu\,(classes\ OO)$,

where $\nu\,(X)$ denotes the number of elements or classes in a set X.

It is easy to see that the consequences of fulfilling the individual sets of conditions (r_1), $(r_1 \wedge r_2)$ and $(r_1 \wedge r_2 \wedge r_3)$ are different. It must also be added that a numeric characterization of the shape of an object or group of objects is far more difficult to carry out. However, some appropriate indices have been proposed in differential geometry.

Summarizing the above, it is worthwhile to note that one of cognitive roles of the $Homm$-tool consists of reducing (synthesizing) all the available information (of primary and secondary importance) about the OO (which is an actual system or process) into a more concise, compact, conceptually capacious, essential, more transparent and workable form. Fundamental questions about what information is more essential with regard to the aim of a particular study are very complex from the point of view of both empirical verification and the choice of selection criteria. There are shared – by theoreticians and model builders – fundamental interdisciplinary principles, such as the Aristotelian principle of non-contradiction, the principle of substantiation and principle of sufficient reason, principle of simplicity and clarity. It suffices to observe the activities of geographers, historians, climatologists to imagine what it means to construct $homm_j \in Homm$. They draw their maps respecting chosen features of the OO, i.e. regions of the Earth, the Earth's surface, mapping chemical and biological characteristics of the Earth or chosen processes of historical events, climatic features for a country, region, continent or the whole Earth. Maps, in their everyday sense, may and will be considered as models of an OO, but appropriate schemes of assigning (adscribing) elements of the $OO \xrightarrow[onto]{into}$ *elements* of the map will be called $Homm$. It means that the creator of a $Homm$ may be considered to be the creator of schemes of models, the CO, that map the OO being studied or the creator of research roads or highways. An ideal CO-map (model) should be identical to the OO.

In economics, sociology and psychology, such models do not exist. It should be mentioned, however, that geographical maps and atlases are useful models in our daily travels. The creators of such maps prepare them with the idea of an ideal map in mind, i.e. a *homm*-map from the point of view of chosen features of the OO.

However, it is possible to consider the following mapping in economics: $homm_T : E_{PL.\ 80-89} \to E_{PL.\ 90-10}$, describing the transformation of the Polish economy of the period 1980–1989 into the Polish economy of the period 1990–2010 as a set of transformations acting on elements of the economy.

It is also instructive to consider an example from the arts. Let us introduce the notation
$OO \equiv$ J. K. Smith from Durham, NC Eva St 2000.01.01,
$CO \equiv$ K. J. Brown's sculpture of the OO made on 2000.01.01.

The statue, the CO of the OO, might preserve shape, but may either shrink or stretch the appropriate dimensions of height, breadth and depth of the OO. The statue, the CO, is treated by observers as a model of the OO. For Mr. K. J. Brown, however, the statue is the OO and Mr. J. K. Smith is a model, i.e. $CO = M$. Note that if the statue, CO, preserves the dimensions and shape of the OO, then it may be treated by observers and mathematicians as an ideal model M of the OO, i.e. $M := CO := OO$ or a concrete isomorphic mapping of the OO onto the $CO := M$. If the observer is a painter, then this $isomm \in \{isomm_j\}_{j \in J}$ is not ideal, since it does not map the colors of the OO, among other necessary features. He will demand that the ideal $isom \in Isomm$ should preserve the color of the hair, skin, eyes, clothes, shoes, and even represent in the face of M a state of emotion in the OO-marble face.

For a chemist or biologist, the marble copy of the OO will not be an ideal $M := CO$, even if it satisfies all the conditions that the painter required for $M := CO$.

This is the case because the living body of J. K. Smith has a completely different chemical structure in comparison to his marble statue copy.

Thus, according to a biologist, the painter's $isomm_p \in Isomm$ is merely a $homm_b \in Homm$, although it is, even for the biologist, a quite acceptable model M of OO.

For a non-professional observer, if the statue preserves the dimensions and shape of the OO, it is often considered by him to be an isomorphic $CO \equiv M$ of the OO, and the whole scheme (process) of the sculptor's actions to be an $isomm_s \in Isomm$.

Note 3.1. In the arts, painters, sculptors, novelists and poets treat real human beings (individuals observed by them) as models for the objects that they create. Later, viewers (or readers) treat these "models" as the OO. Whether, in the eyes of an ordinary viewer, a sculpture is recognized as an *isomm* or *homm* depends on the distance of the viewer from the sculpture and this conclusion may differ from the conviction of the creative artist.

Note 3.2. As already stated – in science, there are two meanings of the term "model". Firstly, by a model M of a given system or process (real or unreal), we understand another system or process (real or unreal), i.e. a CO of the OO. Usually by an "unreal, virtual" system, we understand, e.g. a mathematical representation (or mathematical model) of the OO, i.e. $CO \equiv$ mathematical model M of the OO. If by the OO, we mean a set of one or more mathematical models, then the CO will be another set of mathematical models or another concrete mathematical model. Hence, $CO \equiv$ model (s). In such a case, by $homm_j$ we understand, e.g. a relation between: a model $M \equiv OO$ and a meta-model $MM \equiv CO$, or between models $M \equiv OO$ and meta-models $\equiv CO$, where $homm_j \equiv f_j : M \to MM$. Thus $homm_j \in Homm$ is a mapping between the model M and the meta-model MM. In the case of an isomorphism $isomm_j \in Isomm$, it is an isomorphic mapping of M onto MM.

3.2.2. Formal characterizations of Homm

The term homeomorphism was known before it was introduced into modern mathematics by H. Poincare (1895) in relation to a piecewisely differentiable mapping of subspaces or varieties from $\mathbb{R}^n$. It was also known to A. Möbius (1863) and F. Klein (1872). Further use of this term appeared later in D. Hilbert, L. Brouver, G. Cantor, G. Peano, M. Fréchet, F. Hausdorff, S. Banach, K. Kuratowski, A. Mostowski, W. Sierpiński, A. Tichonow, R. Engelking, A. Grzegorczyk, A. Kostrikin, P. Aleksandrow and other mathematicians and logicians publications. As is well known and stated in §2.1, it is important to make a distinction between a given, concrete homeomorphism $f_j \equiv homm_j$ and a category f of homeomorphisms given by a set $f \equiv \{f_j,\ j \in J\} \equiv Homm_f\,(\cdot)$.

We begin by presenting $homm$ and $Homm$ as a mapping of spaces. The following definition will be very useful.

D1. A homeomorphism (homeomorphic transformation, mapping, correspondence, function, relation) is a given mapping f, such that

$$f : O_1 \to O_2,$$

where $O_1,\ O_2$ are two well defined mapped objects; $\to$ denotes a mapping either into or onto, f is either a given single form of map or given type of forms, i.e.

$$f \equiv f_j or\ f \equiv \{f_j,\ j \in J\}.$$

Note 3.3. We will hereafter understand the object O_1 as a space (e.g. topological, vector, linear, metric or affine), mathematical field, vector lattice, semi-group, group, ring, module, ordered class, algebra, algebraic system, bundle, layer, complex, chain or sheaf. The object O_2 denotes the mathematical object corresponding to O_1. Let us start with situations in which O_1 and O_2 are topological spaces.

D2. For topological spaces O_1 and O_2, the map $f : O_1 \to O_2$ is a homeomorphism between these spaces if f is a bijective (simultaneously injective and surjective) map and both f and its inverse f^{-1} are continuous maps (i.e.$f, f^{-1} \in C^{\circ}$); see: K. Jänich (1984), J. L. Kelly (1957), L. Kantorowitz, G. Akilov (1984), N. Bourbaki (1964), R. Edwards (1965), A. P. Robertson (1964), R. Engelking (1976) and K. Kuratowski (1933).

Note 3.4. From Definition D2, it can be seen that the homeomorphism $f \equiv homm$ maps one topological space O_1 into another topological space O_2. This means that these spaces may be different and that, under D2, O_i does not denote a set of spaces, i.e. that $f : (O_{i1}, \ldots, O_{im}) \to (O_{i1}, \ldots, O_{im})$. If this is the case, then $f = \{f_1, \ldots, f_m\} \equiv Homm$, where $m < \infty$, denotes a homeomorphism category, which is a finite set (sequence, vector) of homeomorphisms. If f is only injective, then we have the case of mapping into, and if f is bijective, then f maps onto.

Note 3.5. When O_i denotes a topological space, then we may choose from the following: Banach spaces, Fréchet spaces, Hausdorff spaces, Hilbert spaces, metric spaces, normed spaces, Tikhonov spaces, Bair spaces or homogenous, vector, anti-discrete, totally regular, extremely non-connective, locally compact,, sequentially, compact, linear, arc-connective, or affine spaces.

Note 3.6. Instead of O_1, O_2, many authors use the symbols denoting the corresponding topological spaces, the symbols X, Y, S, $\tilde{S}$, W, $\tilde{W}$, E_1, E_2, X_1, X_2, $\widetilde{X_1}$, $\widetilde{X_2}$, E, F, G, H, V, W or V_1, V_2. Instead of the symbol f, the symbols F, φ, T, g. Φ, h are also used.

Note 3.7. The following abbreviations of *homm* and *Homm* are also used:

$$O_1 homm_f O_2\ , O_1 hom_f O_2,\ O_1 Hom_f O_2,\ O_1 Homm_f O_2,$$

$$O_1 \left(\underset{f}{\to} O_2\ , O_1 \middle| \underset{f}{\to} O_2, O_1 \underset{f}{\to} O_2,\ O_1 \underset{f}{\to} \right\rangle O_2$$

$$homm\,(f; O_1, O_2)\,,\ hom\,(f; O_1 \to O_2)\,,\ Hom\,(f; O_1 \to O_2)$$

$$\text{or} \quad hom_f\,(O_1, O_2)\,,\ Hom_f\,(O_1, O_2)$$

Note 3.8. From both D1 and D2, a homeomorphism(s) represents(-t) the similarity or great similarity of only pairwisely corresponding objects. However, in the case $f \equiv (f_1, \ldots, f_m)$, we have m such pairs simultaneously,

although these pairs are usually treated separately when checking similarity, due to the technical complexity of simultaneous comparisons.
Rings can also be used as the objects $O_1,\ O_2$, i.e. sets endowed with the operations of multiplication and addition (see: e.g. Kelly (1957), Lang (1970)).

D3. The mapping $f\ :\ O_1 \overset{into}{\longrightarrow} O_2$ of the ring O_1 into the ring O_2 is $homm_f\,(O_1, O_2)$ if f is such that, for $a,\ a' \in O_1, f\,(a+\ a') = f\,(a) + f\,(\ a')\,,\ f\,(1) = \overline{1}$; $f\,(a,\ a') = f\,(a)\,f\,(\ a')\,,\ f\,(0) = \overline{0}$.
The third kind of object to be considered are modules, for which (see Lang (1970)) we have the following definition:

D4. The mapping $f\ :\ O_1 \overset{into}{\longrightarrow} O_2$ is called $O_1 homm_f O_2$, where O_1, O_2 are modules over rings or algebraic modules.
Note 3.9. The concept of an algebraic module was introduced into algebra in the 20-ties of the XX c. to take into account problems regarding algebraic systems, in particular with respect to a map $(a_1, a_2) \longmapsto a_1.a_2$, especially when this map is from K x $G \underset{into}{\longrightarrow} G$, where K is an associative ring with the property of unicity and G is an Abelian group.
The fourth kind of object to be considered are semi-groups or groups.

D5. A map $f\ :\ O_1 \overset{into}{\longrightarrow} O_2$ is called $O_1 homm_f O_2$, where O_1, O_2 are groups or semigroups if $f\,(a_1 a_2) = f\,(a_1)\,f\,(\ a_2)\,\forall a_1, a_2 \in O_1$ transforms 1_{O1} to 1_{O2} , i.e. $1_{O2} = f\,(1_{O1}) = f\,(x)\,f\left(x^{-1}\right)$.
The fifth kind of object represented by O_1 and O_2 are algebras. In this case, we have

D6. A map $f\ :\ O_1 \underset{into}{\longrightarrow} O_2$ is called $O_1 homm_f O_2$, where O_1, O_2 denote corresponding algebras, i.e. algebraic structures with specified operations and elements.

There are also other useful algebraic objects defined in algebra and other fields of pure and applied mathematics, e.g. ideals, lattices, layers, sheafs, clusters, chain complexes. The general scheme of the definition of *homm* remains the same as in D1.

3.2.3. Isomorphisms

In §2.2, it was mentioned that a homeomorphism represents the similarity of the forms of two objects, the *CO* and *OO*. An isomorphism (*Isomm*) represents the equality of the forms of two objects. In practice, the *Isomm* category consists of realistic, natural, documental pictures or images of real things that are treated as approximations of the original object. Mathematicians and logicians treat such representations as approximate isomorphisms. As stated in §3.2.1, in reality, an ideal $isomm \in Isomm$ creates an exact copy of the original object, *OO*, i.e. a 1–1(one to one) mapping of, e.g. $house_1$ onto $house_2$. Children's toys (e.g. cars, tanks, dolls, houses, model railways) are good examples of approximately isomorphic copies of objects, *CO*s, (with regard to the original objects, *OO*s, (i.e. real cars, tanks, girls, houses, railways), where the *CO*s are shrunk versions of the *OO*s. While discussing the concepts of isomm, homm, it was underlined that they need to be generalized in content or renamed. It was stated that there is a fundamental need to take into account features other than the form. These other characteristics might concern e.g. the chemical, physical and biological components of the macrostructure of the elements of the *CO* and *OO*, as well as the microstructure of each element responsible for color, smell, sound, taste, tactile qualities of objects, and opportunities of connecting and disjoining particles, parts and elements into live attractive forms. However, from the point of view of constructing an isomorphism or homeomorphism, enriching features is not good practice, due to the ambiguity of such a concept. A better approach is to define appropriate forms of maps and study their research utility.

Thus, we may introduce

isometrism – isometronism – as a relation – when all the dimensions of the *CO* and *OO* are equal or as a map between the *CO* and *OO* that preserves all the dimensions of the *OO*,

isochorism – a relation (map) that preserves equality of the volume (surface area) of the *CO* and *OO*,

isochronism – a relation (map) that preserves equality of the period of motion of the *CO* and *OO*,

isomerism – a relation (map) that preserves the spatial structure of the corresponding elements of the *CO* and *OO*,

isostasism – a relation (map) that preserves equality of the equilibrium states of the *CO* and *OO*,

isotachism – a map that preserves equality of the velocities of the *CO* and *OO*,

isopitanotetism – a map that preserves equality of the probability of the *CO* and *OO* being in corresponding states,

idiodiastemism – a map that preserves equality of the distance between:

- CO_t & OO_t (where CO_t denotes the copy of an object at time t, etc.), or
- the centers of CO_t & OO_t, or
- the periodic mean center of CO_t & OO_t, or
- the periodic median center of CO_t & OO_t.

All these "isoisms" can be conjugated (coupled, tripled, ...) with the concept of isomorphism. It is an open question as to what kind of consequences such conjugation would bring about in theory and practice. It is obvious that finding these consequences is a very attractive aim, not only for practitioners in many applied fields of science.

When defining an isomorphism – mathematicians most often use such topological terms as: "topological space", "metric space", "vector space", "linear vector space". Some authors simply use "linear space" or "space". The following terms are also relatively common: "group", "ring", "algebra", "algebraic system", "module", "ideal", "body", "category", "lattice".

Logicians prefer using the terms "objects", "relational systems" or "mathematical field".

Respecting the approach of going from the general to the particular, we start with a convenient logical description of *isomm.* By the field (campus) of the relation R, we mean $C(R) = D(R) \cup CD(R)$, where D denotes the domain and CD is the co-domain. Suppose that there are two relations R_1 and R_2, and the set of reciprocally unique relations $\{R_{1\to 1}\}$. It will be said that (see, e.g. L. Borkowski (1977)),

$$R_1 isomm_R R_2 \equiv \exists R \in \{R_{1\to 1}\} :$$

$$\left[D(R) = C(R_1) \wedge CD(R) = C(R_2) \wedge \forall x,\ y : \left(R' x R_1 R' y = x R_2 y\right)\right],$$

i.e. the pair of relations $(R_1,\ R_2)$ are isomorphic if there exists such a reciprocally unique relation R that assigns the domain of R_1 to the domain of R_2. If R_1 is isomorphic to R_2, then the cardinality of $C(R_i)$ satisfies $\nu(C(R_1)\ ) = \nu(C(R_2)\ )$, and if $R:\ R_1 isomm_R onto\ R_2$, then (R_1, R_2) are relationally identical. This property is often used when preparing geographical maps or diagrams and engineering plans. The structural position of the elements in the domain is also preserved.

A more general presentation of *isomm* concerns ordered systems of $(m+1)$-tuples. Given the systems $S_1 \equiv < A; R_1, \ldots, R_m >, S_2 \equiv < B; R'_1, \ldots, R'_m >$, where $A,\ B$ denote the sets of relations $\{R_1\}$ and $\{R'_1\}$, respectively and preserving the ordering, we may define

$$S_1 Isomm_R S_2 \equiv A \sim_R B \wedge R_1 Isomm_R R'_1 \wedge \cdots \wedge R_m Isomm_R R'_m,$$

$\equiv \exists R : S_1 Isomm_R S_2$, where $\sim R$ denotes the isocardinality of the elements of the sets A and B.

Note 3.10. In $S_1,\ R_1$ may denote a specific function (map, transformation, operator, correspondence) f_i belonging to a chosen class of functions, and the set A may denote a given space, group, ring, algebra, algebraic system, module, ideal, lattice, category, body or field.

Moreover $isomm_R \equiv \varphi$ may itself also be a function (map, transformation, operator, correspondence, operation) that maps S_1 onto S_2.

It is also possible to distingnish other representations of $isomm,\ Isomm$.

Some of them are given below:

$$S_{10} \equiv \left\langle A;\ \{f_i\}_1^{m_1},\ \{\alpha_j\}_1^{m_2}, \{R_k\}_1^{m_3} \right\rangle,$$

$$S_{20} \equiv \left\langle B; \left\{ f_i^{'} \right\}, \left\{ \alpha_j^{'} \right\}, \left\{ R_k^{'} \right\} \right\rangle,\ \sum m_i = m,$$

$$S_{11} \equiv \langle A; f_1, f_2, \ldots, f_{m-1}, R_m \rangle,$$

$$S_{21} \equiv \left\langle B; f_1^{'}, f_2^{'}, \ldots, f_{m-1}^{'},\ R_m^{'} \right\rangle,$$

$$S_{12} \equiv \langle A; f_1, f_2, \ldots, f_{m-2}, P\left(R_{m-1}\right), R_m \rangle$$

$$S_{22} \equiv \left\langle B; f_1^{'}, f_2^{'}, \ldots, f_{m-2}^{'},\ P\left(R_{m-1}^{'}\right), R_m^{'} \right\rangle,$$

$$S_{13} \equiv \langle A; f_1, f_2, \ldots, f_{m-2}, P\left(R_{m-1}\right),\ P\left(R_m\right) \rangle,$$

$$S_{23} \equiv \left\langle B; f_1^{'}, f_2^{'}, \ldots, f_{m-2}^{'},\ P\left(R_{m-1}^{'}\right), P\left(R_m^{'}\right) \right\rangle,$$

$$S_{14} \equiv \langle A; f_1, f_2, \ldots, f_m, \rangle,\ S_{24} \equiv \left\langle B; f_1^{'}, \ldots, f_m^{'} \right\rangle.$$

For these systems of objects, we have the following definitions of isomorphisms:

$S_{1i} Isomm_\varphi S_{2i} \equiv \exists \varphi : S_{1i} \overset{onto}{\longrightarrow} S_{2i}$, where φ is mutually unique bijective map with respect to $\{f_i\}$, $\{P\left(R_j\right)\}$, $\left\{P\left(R_j^{'}\right)\right\}$, R_m, $R_m^{'}$ and $A,\ B$ preserve the operations defined for the objects being mapped, where $i = \overline{0,4}$.

In particular cases, these general schemes defining *isomm* or *Isomm*, have particular forms.

J. Dieudonné (1968) takes S_{1i} and S_{2i} to be mathematical structures in which A and B are groups, rings, vector spaces or Euclidean spaces. For them, $isomm \equiv \varphi$ is a bijective mutually unique map. For linear spaces A and B, Postnikov (1979) considers $\varphi : A \xrightarrow{onto} B$ as *isomm* when φ is a bijective map.

I. N. Bronschtein et al. (2009) consider the bijective map $\varphi : A \xrightarrow{onto} B$ as an isomorphism of algebraic groups or vector spaces A and B.

In his monography on the representations of groups, M. A. Naimark (1955, 1976) gives many definitions and statements regarding isomorphisms of groups, rings, topological spaces and topological groups.

C. Kosniowski (1980) and A. Császár (1978), e.g., prefer to treat *Isomm* as simultaneous monomorphisms and epimorphisms.

B. L. Van der Waerden (1967) defines an isomorphism between groups as a reciprocally unique bijection φ that preserves the operation of multiplication within groups.

K. Jänich (1984) considers *isomm* as a bijective map $\varphi : A \underset{onto}{\longrightarrow} B$ when the corresponding $f^{-1} \in \mathbf{L}(A, B)$ belongs to the set of linear maps, and A, B are given topological spaces. He also gives very instructive examples of *homms* and *isomms* (e.g. the map $\varphi : t \to e^{it}$, which is a bijection, but it is not *homm*.

In an algebraic compendium, S. Lang (1970) gives a broad, clear overview of *Homm* and *Isomm* categories defined when (A, B) are groups, semigroups, factor groups, rings or vector spaces.

D. Przeworska-Rolewicz (1988) defines $isomm\,(\varphi; A, B)$ as a one-to-one map φ that preserves the ring's operations of addition and multiplication, when the rings A and B are commutative – with zero and unit elements belonging to both A and B.

One interesting possible application gives an order isomorphism. In W. Archangelski and W. Ponomarev (1974), e.g. we find that the similarity ordering *isomm* is a map $\varphi\,(A, <) \xrightarrow{onto} (B, \prec)$ that is monotonic and mutually unique (bijective), i.e.

$\forall a_1, a_2 \in A : a_1 < a_2 \to \varphi\,(a_1) \prec \varphi\,(a_2) \wedge f\,(A) = B \wedge \varphi^{-1}$ is monotonic.

Using the sets of linear operators $\mathbf{L}(A, B) \equiv \{\varphi\}$, S. S. Kutateladze (1983) defines an algebraic *isomm* φ as a linear operator $\varphi \in \mathbf{L}(A, B)$ that has a linear inverse $\varphi^{-1} \in \mathbf{L}(A, B)$, where A and B are vector spaces, and $\varphi : A \xrightarrow{onto} B$.

A. Malcew (1970, 1975) defines an *isomm* $\varphi : A \to B$, where A and B are linearly ordered sets, algebraic systems, compositions of algebraic systems, algebras, modules, rings or spaces: affine, unitary, metric or linear. For him, the arrow denotes a map either into or onto depending on the assumption of whether $\varphi(A) \neq B$ or not.

A. I. Kostrikin (1977) considers *isomm* $\varphi : A \to B$, where A and B are groups or algebras.

Uniform isomorphisms, discussed by J. L. Kelley (1957), are more complex constructions. For two topological spaces A and B, a map $\varphi : A \overset{onto}{\longrightarrow} B$ is uniformly isomorphic if $\varphi, \varphi^{-1} \in C^{\circ}$ and φ is a reciprocally unique map.

In algebraic geometry (see, e.g. F. Hirzebruch (1966)), A and B may be bundles over a given topological space. In this case, isomorphisms can be defined with respect to strata around chosen points.

Some authors (e.g. R. Edwards (1965)) define A and B to be spaces, in particular topological vector spaces, and *isomm* φ can even be defined as a linear bijective map of factor space A into an appropriate space B.

One can also define *isomm* φ as $A \to B$, where A and B are appropriate sets of points of attraction, sets of points of repulsion – i.e. concepts that are vital in the analysis of dynamic systems, models of dynamic systems, including stochastic models of dynamic systems.

For stochastic models or systems, the concept of a measurable space is fundamental. A. Kolmogorov (1933) used Boolean algebra and the terminology of Borel sets in defining a probability space.

Let $(\Omega, F_\Omega) \equiv A$, $(R, F_R) \equiv B$, where Ω denotes the set of elementary events for a random trial, F_Ω the Borelian field of all subsets of Ω, R is the set of real numbers, and F_R the Borelian field of all subsets of R. Then *isomm* $\varphi : A \to B$ will be a map from the measurable space A to the measurable space B. By probability spaces, he understands the spaces: $S_1 \equiv (\Omega, F_\Omega, P_1)$, $S_2 \equiv (R, F_R, P_2)$, where P_1 denotes a probability measure determined on the Borelian field F_Ω and P_2 is a probability measure defined on the Borelian field F_R, where $P_1(\Omega) = 1 = P_2(R)$.

J. Neveu (1970) calls the spaces S_1 and S_2 probability spaces. The random variable X may be defined as either

$$X : S_1 \to S_2$$

or $X : (\Omega, F_\Omega) \to (R, F_R)$ or $X : \Omega \to R$.

A. Bulinski and A. N. Shiryaev (2003) define an isomorphism between $S_{20} \equiv (R, F_R)$ and $S_{21} \equiv (R_{01}, F_{R_{01}})$, where $R_{01} \equiv [0, 1]$, and $F_{R_{01}}$ is the Borelian family of all subsets of the line segment $(0, 1)$.

Thus $\varphi : S_{20} \to S_{21}$ is an isomorphism if φ is bijective. By introducing more specific spaces and the random variable X which has a normal distribution function $F_X \equiv P_2 = \Phi_\mathrm{N}$, we sieve the space S_2 into $S_{21} \equiv (R, F_R, \Phi_\mathrm{N})$, in particular when F_R does not cover all the subsets of R. In this case, the transformation $\varphi : S_2 \to S_{21}$ is an isomorphism when φ is a bijective map of the space S_2 onto the space S_{21} which preserves probabilities.

The above discussion of *homm* and *isomm* maps indicates that there also exists the possibility of treating the random variable X as a *homm* or *isomm* map between the spaces: (Ω, F_R) and (R, F_R), S_1 and S_2, when these spaces fulfill the appropriate conditions introduced in the above definitions of the *homm* and *isomm* maps.

3.3. Homeomorphism, isomorphism and models

It is difficult to overvalue the role of homeomorphisms in theory and practice.

Firstly, they enable us to replace complex mathematical objects (in particular, models of real systems) by simpler ones.

Secondly, they simplify the classification of real objects, their mathematical descriptions in the form of models. Thirdly, they make it easier to find crucial properties of models, as well as methods for analyzing them.

Fourthly, since homeomorphisms preserve important properties of the systems studied (in particular, orbits), including the properties of finiteness and commutatitivity, their role in determining the operational principles of modeling is also crucial. *Homm* maps preserve the qualitative properties of specific sets and spaces such as spheres, perfect sets, varieties, sheaves and bundles.

Isomorphisms preserve properties more exactly and fully than homeomorphisms. Their power to preserve is either ideal or, in practice, should be extremely close to one-to-one.

Before we consider some relations between the *Homm*, *Isomm* and *Model* categories, it is necessary to recall the hierarchy of objects studied in science and practice. We describe the hierarchy in scientific research using a pyramidal structure. This pyramid can be illustrated as follows:

CO_{IV}: meta – models *Homm*, *Isomm*

CO_{III} : descriptions of objects, OO, CO_I and CO_{II}, in the language of mathematics and logic (e.g. sets, relations, maps, models)

CO_{II} : description of objects, OO and CO_I, in a natural language

CO_I : actual man-made copies of OO (e.g. copies of precious stones, bio-organs, robots, cars, artistic creations, TV-productions

OO : physical bodies, bio-organisms, the Universe, actual facts, events

Fig. 1. Hierarchy of objects in research.

In this text, we mainly deal with the CO_{III} and CO_{IV} levels of this pyramidal architecture of cognition.

In order to pass on more smoothly to models, we shall use various concepts of functional spaces. Linear spaces with norms are examples of functional spaces. Other examples of functions spaces are:

s1) $X \equiv \left\{x \in C^{\circ}\left[0,1\right]\ x\left(t\right), t \in \left[0,1\right], \left\|\cdot\right\|\right\}$, the space of functions x, which are continuous on $[0,1]$;

s2) $X \equiv \left\{x\left(t\right) \equiv t;\ t = t_1, \ldots t_n; \left\|x\right\|_p = \left(\sum_{k=1}^{n} \left|t_k\right|^p\right)^{1/p}\right\}$ is an n-dimensional space l_n^p, $p \geq 1$, where the elements of the space are systems of n real or complex numbers;

s3) l^{∞} spaces of infinite sequences of bounded real or complex numbers with $x\left(t_k\right) = t_k$ and norm $\left\|x\right\| = sup_k\left|t_k\right|$;

s4) the space $C\left(\Omega\right)$ of $x \in C^{\circ}\left[\Omega\right]$, where Ω is a compact Hausdorff topological space, the values of the x –function belong to R or C, and the norm is given by $\left\|x\right\| = sup_{t\in\Omega}\ \ \left|x\left(t\right)\right| < \infty$;

s5) $L^p\left(\Omega, \sum, \mu\right)$ – the space of linear Banach functions which are degree – p integrable, $p \geq 1$, where Ω is a set, $\sum$ is the σ–algebra of the subsets of Ω, μ – a measure in $\sum$, so the functions x are $\sum$ – measurable, such that $\int_{\Omega}\ \left|x\left(t\right)\right|^p d\mu < \infty$ and $\left\|x\right\| = \left(\int_{\Omega}\ \left|x\left(t\right)\right|^p d\mu\right)^{1/p}$;

s6) $L^P\left(R^n, F_{R^n}, P_{\mathbf{Y}}\right)$, where $L^P\left(\Omega, F_{\Omega}, P\right)$ is analogous to (s5), $P_{\mathbf{Y}}$ the probability measure of a vector random process $\mathbf{Y}$, F_{R^n} the σ - algebra of the subsets of the set R^n, Ω the set of elementary events, F_{Ω} the Borellian σ-field of the subsets of Ω, P a probability measure on F_{Ω}, $\mathbf{Y} : L^P\left(\Omega, F_{\Omega}, P\right) \to L^P\left(R^n, F_{R^n}, P_{\mathbf{Y}}\right)$;

s7) the Schwartz space of tempered distributions T, i.e. $D\left(\Omega_{\mathrm{k}}\right) \to D_H$ is a space $D'\left(\Omega_{\mathrm{k}}\right)$ conjugate to the space $D\left(\Omega_{\mathrm{k}}\right)$, where $val\left(T\right) \subset C^1$, Ω_{k} is an open region of the space E_{k}, $D\left(\Omega_k\right)$ is a set with pseudotopology $C_\circ^\infty\left(\Omega_k\right)$, the functions $\varphi_i \in D\left(\Omega_k\right)$, D_H is a Hilbert space, T is a continuous, linear map of, e.g. the form: $\langle\varphi,\mu\rangle \equiv \int_{\Omega_{\mathrm{k}}} \varphi\left(x\right) d\mu\left(x\right), \varphi \in C_\circ^\infty\left(\Omega_k\right),\ \mu \equiv T$, $\langle\varphi,\delta_x\rangle \equiv \varphi\left(x\right),\ \ \varphi \in C_\circ^\infty\left(\Omega_k\right),\ \delta_x \equiv \delta\left(x - x'\right)$ is the Dirac distribution, $\langle\varphi,T\rangle \equiv \int_{\Omega_{\mathrm{k}}} \varphi\left(x\right)\overline{T\left(x\right)}dx$, T is measurable and locally integrable with respect to the measure dx.

s8) X is a modular Orlicz space of functions.

3.3.1. Category of model

Based on the above, we may approach the category of model. Etymologically, the English word "model" originates from the Latin "modulus", which meant "measure, pattern, formula" and passed into Italian as "modello", as well as into French as "modéle". Commonly, it is used in the sense of imitating or mimicking a chosen object. In the arts, architecture and engineering it is either an imitation of a real or virtual object, often of smaller size, or an artistic creation based on some object. Hence, some works of artists may be thought of as models of some mysterious vision of a virtual OO. It is obvious that a model is a pattern or formula for something (usually an OO, CO_I or CO_{II}). Such objects have creators (either non-human nature or human(s)) and are created at a specific time and place. Discussions and generalizations regarding the category of model can be carried out at three levels of inference.

At the first level – replacing the heuristic idea of equivalence, M equiv V, to the object O by the relation M simil O or M close to O, where "simil" denotes similar to and "close" denotes the closeness of M to O is a weakening of the concept of a model M. However, similarity or closeness is a matter of degree and may be defined in various ways (topologically, algebraically, logically). Thus (see J. Gastev) M equiv O requires the transition property, but M simil O or M close O does not. This means that the meaning of M in this sense is more flexible. Such a sense preserves the properties of symmetry and reflexivity of M and O, but does not preserve the property of transitivity. Therefore, if transitivity does not enter into the relation "to be a M", since it is not postulated, but is required by the mathematical definitions of *homm*, *isomm*, then this generalisation means that the concept of closeness or similarity differs from the mathematical meaning of *homm*, *isomm*.

At the second level, a generalization of the concept of a model M may be formulating M in a way that preserves certain relations satisfied by the definitions of *homm* and *isomm*. Hence, M may reflect generalizations of the relations of symmetry, transitivity, closeness, similarity and heritage. (J. Gastev, e.g. includes generalization via a modification of the property of symmetry).

At the third level, generalizations of a model may be introduced by considering the theory of fuzzy or rough sets (see L. Zadeh, Z. Pawlak, B. Birjukov) instead of classical set theory or topology. Hence, a model M may be usefully treated as a relation between objects O_1 and O_2, i.e. O_1 model O_2 by performing a kind of object identification in research situations involving fuzzy observation of real life and time. Usually $O_2 \equiv CO_I, CO_{II}$, and $O_1 \equiv OO$.

Therefore, a model M may reflect any similarities or simplifications which occur not only in Platonic-type ontologies. The author shares the view that M is not only an objective attribute of O, but also a product of our (in particular, researchers') modeling practices, experiences, subjective consciousness and view of the world, as well as, subconsciously, the product of the degree of dependence, or independence, between our current life, sources of finance for research and groups of influence (also within research circles).

Since homoios means similar, we may treat $homm - s$, $Homm - s$ as models or categories of models which preserve the similarity between OO, CO_I, CO_{II} according to a given definition (see §2).

There are many criteria for classifying scientific models. One of them is the purpose for which a model will be used. Here, we distinguish between models for the purpose of simulation and other models. Simulations enable us to learn about unknown features of OO, CO_I, CO_{II}, or to acquire certain necessary skills in various professions (e.g. pilots, astronauts). When the goal is to develop a method of reasoning and explain phenomena, then we talk about either deductive-nomological models of phenomena and events (C. Hempel, P. Oppenheim), deductive-statistical models (C. Hemple), deductive-economic or deductive-econometrical models (B. Stigum), inductive-statistical models (D. Dempster, H. Reichenbach), causal models (E. Nagel), semantic models, syntactic models, mathematical models, meta-mathematical models, iconical models, potential models, proper models, language models, empirical versus theoretical models, geometric models, models of economic growth, models of the physical world, standard versus non-standard models, formal versus non-formal models, homogenous or non-homogenous models, saturated or non-saturated models, universal or non-universal models, macro or micro-state models, stable or unstable models, computable or uncomputable models, simple or complex models, rigid or flexible models, small or large models, short-run or long-run models, models of open or closed systems, linear or nonlinear models, deterministic or stochastic models, learning or self-learning models, neural-nets or evolutionary models.

Many interesting comments on some of these types of models are given in J. Barwise (ed.) (1977), A. Tarski (1954), H. J. Keisler (1971), A. Malcew (1970), J. Woleński (2005), A. Grobler (2006) and J. Gastev (2009).

A. Tarski (1936) introduced the concept of a "model" with respect to the concept of a "sentence".

By a model (English translations also use the term "realization") of a class L of sentences, we shall mean any sequence of subjects or objects which fulfills each sentence function (formula) from the class $L^{'}$. In this sense, he proposes to speak about models of systems of axioms in a deductive theory, i.e. about the axiomatic part of such a theory. Hence, a sentence S_i is implied logically by sentences $\{S_{ax,j}\} \equiv A$ from the axiomatic part of a theory T iff each model M_A of class A is also a model of the sentence S_i. Thus, there is no concept of a model if the subjects do not satisfy the sentence, e.g. replacing the constants 2, 3, 5 in the sentence $2 + 3 = 5$ by variables x, y, z, we transform the sentence $2 + 3 = 5$ (belonging to the class L) into the sentence function $x + y = z$ (belonging to the class L' of sentence functions).

Thus, for A. Tarski, any sequence $\{x, y, z\}$, $\{a, b, c\}$, $\{p, q, r\}$ fulfilling $x+y = z$, $a+b = c$, $p+q = r$ is a model of class $L \equiv \{1+2 = 3, 2+3 = 5, \dots, 2^2 + 2^3 = 2^2 \cdot 3, \dots\}$ sentences in a formalized language, e.g. the arithmetic of natural numbers. This means that he speaks about models not in the sense of functions, systems of functions or maps, but in the sense of a set of subjects, which are formally, logically and operationally similar in their role in sentence formulas.

Therefore, his understanding of a model is here set-theoretic, not in terms of relational systems, functional analysis or the theory of topological mappings.

A. Tarski's second concept of a model (1953) links the concepts of a theory and of being logically valid to general (important) sentences. According to him, a sentence S_j is logically valid (provable), if it is derivable from logical axioms or it is fulfilled by a given or all realizations from a given class of realizations under a formalized theory T which is decidable. The sentence S_j is logically true, if it is fulfilled for each possible realization of the theory T. The theory T consists of a set A of logical axioms, inference operations (of which modus ponens is the most important), non-logical constants $\mathbf{C} = \langle C_o, \dots, C_n, \dots \rangle$, logical constants $\Gamma = \langle \gamma_o, \dots, \gamma_m, \dots \rangle$, where γ_i may be e.g. the sign of a logical implication or a quantifier, C_i denotes a predicate (relation), operation (function), given individuum, or one or more members of predicates and their possible logical consequences. The sentence S_j is a consequence of the set A if it is fulfilled in T_R, i.e. by all possible realizations of the theory T, where $T_R \equiv \langle U, C_o, \ldots, C_n, \ldots \rangle$, U is a universum of the system T_R. A. Tarski calls any possible realization T_R of T for which all of the valid sentences of T are fulfilled in T_R a model M_T of the theory T.

We shall use the following definitions:

DT. By a theory, we mean

$$T \equiv \langle A, \Gamma, C, S_\theta, S_{\neg\theta} \rangle ,$$

where A, Γ, C are defined as above, S_θ is a set of T-theses, $S_{\neg\theta}$ is a set of sentences in the theory T, which are not treated as theses.

DRT. By a realization T_R of the theory T, we will mean

$$T_R \equiv \langle A, \Gamma, C_R, S_{\theta,R}, S_{\neg\theta,R} \rangle ,$$

where $C_R, S_{\theta,R}, S_{\neg\theta,R}$ are concrete realisation-specific forms of $C, S_\theta, S_{\neg\theta}$.

DMT. By a model M_T of the theory T, we mean $M_T \equiv M_{T_R} \equiv T_R$ for which

Version a) all $S_{\theta,R}$ are fulfilled in T_R,

Version b) all $S_{\theta,R}$ are fulfilled in T,

Version c) all $S_{\theta,R}$ & $S_{\neg\theta,R}$ are fulfilled in T_R,

Version d) all $S_{\theta,R}$ & $S_{\neg\theta,R}$ are fulfilled in T.

Definition DMT shows that the concept of a model is not unique. The versions (a)–(d) are not disjoint.

The following sentences may be included in the set $S_{\theta,R}$:

Ex 1. Consider linear statistical models M_0, M_1 based on statistical theory T, where the model M_1 is more reductive and

$$M_{T0} \equiv \langle A\ , \Gamma, C_p,\ \mathbf{R}^{n\times k} \sum, \sum_R, \mathbf{Y} = \mathbf{X}\beta + \varepsilon,$$

$$E\varepsilon = 0, D\varepsilon = \Omega, \mathbf{X} \in R^{n\times k}, \beta \in R^k, \sum = (\Omega, F_\Omega, P)\ ,$$

$\sum_R = (R^n, F_{R^n}, P_Y)$ denotes definitions, lemmas and theorems, respectively, concerning functions defined on elements of $M_{T0}, M_{T1} >$;

$$M_{T1} \equiv M_0 \wedge \varepsilon \sim N(\mathbf{0}, \Omega) \wedge \nu_{\mathbf{X}} \geq 100,$$

where: ε denotes a random vector, $N-$ the gaussion distribution, E–the expectation operation, D–the variance–covariance operation, R^{nxk}–the set of matrices of real numbers $\mathbf{X}$ with n rows and k columns, $\mathbf{Y} : \sum \rightarrow \sum_R$ a mapping of the probability space $\sum$ into the probability space $\sum_R$, Ω a set of elementary events, F_Ω (or F_{R^u}) is the Borelian-field of the subsets of Ω (or F_{R^n}), $P\,(\text{or}P_Y)$ a probability measure on $F_\Omega\,(\text{or } F_{R^n})$; $\nu_{\mathbf{X}} = \sqrt{\lambda_{max}(\mathbf{X}^T\mathbf{X}) / \lambda_{min}(\mathbf{X}^T\mathbf{X})}$, $\lambda_{max}\,(\lambda_{min})$ are the maximal (minimal) eigenvalues of the matrix $\mathbf{X}^T\mathbf{X}$. Note that under M_1, $P_Y = F_Y$, where F_Y is the distribution function of the normal probability distribution.

Ex 2. (the case of a mathematical model of space-dependent, multi-species interactions)
$M_{T0} \equiv \left\langle N_\alpha, \sum, \sum_R, \dot{\mathbf{N}}(t) = g(\alpha, \nabla\mathbf{N}, \mathbf{V}, \mathbf{N}, x, y, t, \alpha, \beta, \gamma, k, \varepsilon),\right.$ $E\varepsilon = \mathbf{0}$, $D\varepsilon = \Omega$; $k, \alpha, \beta \in R^2$, $t \in R^+$, $x, y \in R)$, $\sum, \sum_R$ and further elements as in M_{T0} in Ex 1$\rangle$,
where: N_a is the set of natural numbers; α, β, γ, k are constant at time t, and $g(\cdot)$ is as defined below:

$$\frac{\partial N_1(t)}{\partial t} = div\,(\alpha_1 \nabla N_1(t)) - div\,(V_1(t)\,N_1(t)) - \sigma_1(x,y,t)\,N_1(t) +$$

$$+\beta_1 N_1(t)\left(1 - \frac{N_1(t)}{k_1}\right) - \gamma_1 N_1(t)\,N_2(t) + \varepsilon_1(t)$$

,

$$\frac{\partial N_2(t)}{\partial t} = div\,(\alpha_2 \nabla N_2(t)) - div\,(V_2(t)\,N_2(t)) - \sigma_2(x,y,t)\,N_2(t) +$$

$$+\beta_2 N_2(t)\left(1 - \frac{N_2(t)}{k_2}\right) + \gamma_2 N_1(t)\,N_2(t) + \varepsilon_2(t),$$

where $N_i(t)$ denotes the number of prey $(i = 1)$ or predators $(i = 2)$ at time t, in a given region of space, $div\,(\alpha_1 \nabla N_1(t))$ denotes the level of dispersion of the population, $div\,(V_i(t)\,N_i(t))$ is the displacement of a population in space, $\sigma_i(\cdot) \cdot N_i(t)$ denotes the induced mortality rate of individuals in species "i", β_i stands for the growth rate of the population of species $i, i = 1, 2$, and γ_i is the predation rate, which affects the population sizes, N_1, N_2, of the species of prey and predators, ε_i describes a random disturbance in the speed of changes in the numbers of prey and predators caused by factors not included in the model (for a biological motivation of this specification, see the paper of R. C. Sossae, et al, 1999).

$M_{T1} \equiv \left\langle N_a, \sum, \sum_R, \dot{\mathbf{N}}(t) = g \right.$ (as in M_{T0} except that $(1 - N_i(t)/k_i)$ is replaced by $\left(\overline{N}_i - N_i(t)/k_i\right)$ and $(E\varepsilon = 0, D\varepsilon = \Omega)$ by $\varepsilon \sim N(0, \Omega)$), otherwise the form of the model is the same as for M_{T0} in Ex. 1$\rangle$.

Similarly to Ex. 1, the model M_{T1} from Ex. 2 is more reductive than M_{T0} in Ex. 2 "will be" our sentence $S_{j,\theta} \in S_{\theta,R}$.

Note 3.11. Models of capital markets and a discussion of their relations were presented by W. Milo in W. Milo, et. al (2010).

Ex 3. (J. R. Lagrange's type models of the motion of one body)

$$M_{T0} \equiv \left\langle R^3, A, C^2, m\ddot{\mathbf{r}}{=}\mathbf{F}{+}\mathbf{Z}, \right.$$

$$\left. \mathbf{Z} = \sum_{i=1}^{2} \lambda_i \nabla g_i(\cdot), g_i(x, y, z, t) = 0, i = 1, 2, g_i \in C^2, t \in R^+ \right\rangle,$$

where A denotes the set of Galileo-Newton axioms of motion,

C^2 – is the metric space of scalar, twice differentiable functions $g_i()$,

R^3 – three-dimensional Euclidean space,

m – constant mass of the body,

$\ddot{\mathbf{r}}{=}\partial^2\mathbf{r}(t)/\partial t^2$, $\mathbf{r}(t) = (x(t), y(t), z(t))^T$– is the vector denoting the spatial location (position) of the body,

$\mathbf{F}$ – total force acting on the center of mass of the body,

$\mathbf{Z}$ – force of reaction equilibrating the body,

$$g_i(\cdot) = grad\ g_i(\cdot) = (\partial g_i/\partial x, \partial g_i/\partial y, \partial g_i/\partial z)^T,$$

$\lambda_i \in R$ – Lagrange multiplier, $i = 1, 2$ (constant with respect to time)

$$M_{T1} \equiv \left\langle R^3, A, C^2, m\ddot{\mathbf{r}}(t) {=} \mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t) \in B(0,1), t) \right.$$

$$\left. +\mathbf{Z}\left(\mathbf{r}, t, g_i(\mathbf{r}(t)) \in B(0, 1/2), g_i \in C^2, t \in R^+\right)\right\rangle,$$

where

$$\dot{\mathbf{r}}(t) {=} \partial\mathbf{r}(t)/\partial t,$$

$B(0, 1)$ denotes a ball with center at 0 and radius equal to 1.

It can easily be seen that the model M_{T1} is a reduction of the model M_{T0}.

3.4. Principles of and problems related to modeling

When considered as mathematical structures, models are similar in their status to the relation between musical notes (signs, symbols) in a musical score to the recreation of music by an artist or group of artists. The structure of the musical score is only an approximation of a performance of live music. It is important to underline that models can be used to study the functioning of real world systems, processes, events and courses of events. By "to model", we usually mean to imitate, pattern, shape, describe or forecast.

R. Thom (1977) and M. Heller (2006) clearly distinguish between various attitudes to modeling. One approach, seen in the works of Gallileo Gallilei and I. Newton, is pragmatic and consists of formulating computable models of physical phenomena. The second is shared by Cartesius and Leibnitz and consists of efforts aimed at explaining phenomena, their ontological essence and describing fundamental ideas and their relations. For R. Thom, the spectacle of the Universe lies in an unceasing motion of the births, development and destruction of various forms of objects. Scientists usually assume that the form of objects is relatively durable. Concepts of models are needed to analyze the behavior of such objects in space and time. Here, a model is meant as a way of selecting such shapes that are relatively durable in space and time from the "sea of forms". Such models capture the shape of the object modeled, but do not capture its substantiality, individuality or structure. For a formalizable phenomenological process P, if we can effectively define a bijective map Φ – a mapping of the formal system F for evaluating forms into its original counterpart, then we may say that we have built a formal model based on a map Φ. The map Φ has to fulfill two conditions: 1) each state S_a of F_a may be parameterized by the set $\{s_a\}$ of sentences of the formal system F; 2) if the state evolves from $S_a \rightarrow S_b$, then the state S_b may be described by a set $\{s_b\}$ of sentences of the formal system F_b, where $\{s_b\}$ can be deductively inferred from $\{s_a\}$ within F_b. So P, F, Φ are such that Φ^{-1} transforms a time sequence of results in P into a logical result of F. Such a model can thus translate a succession of states in a time sequence of forms of objects into logical inference from sentence sets within a given formal system, i.e. model M. According to such a model, we have more than Hume's succession of phenomena. Such a model does not assume determinism. Many consequences can follow from $\{s_a\}$, e.g. $\{s_b\}_1, \ldots, \{s_b\}_m$, and there can be multiple predictions from a model.

In any formal model, there should be kinematic and dynamic parts. The description of the kinematic part should cover the states of P using sentences from the formal system. The description of the dynamic part (evolution of the state) should use implication sentences from within the formal system.

When using multi-valued systems of logic the description of the dynamics should use sentences describing the probabilities of transitions from states $\{S_a\} \rightarrow \{S_b\}$. In the case where all these probabilities are equal to 1, these are deterministic models. The principle of Leucipos and Democrit that "no matter ever originates without a cause, but always do from a sufficient and necessary condition" and the axiom that "the only real things are atoms and vacuums" are successfully used in the modern physics of elementary particles, as can be seen in successes in: technology, computational tools, experimental methods and tools, mathematics, logical calculus, the natural sciences and modeling. These successes lie in the use of formal continuous and discrete models of P–processes. Modern physics, chemistry, biology the engineering sciences and interdisciplinary science are full of theoretical and practical models. The social sciences, economics, and the philosophy of science also use a wide range of models as tools of analysis. In these fields, models are used either as tools for describing and explaining phenomena, or as tools for prediction and decision making. Some modelers use them as tools for analyzing processes. Other users may use them more often as tools for forecasting or as descriptive and explanatory tools.

The principles of modeling imply the necessity of selecting appropriate models. On the other hand, given models require selecting the corresponding principles of modeling. In turn, the choice of a model depends on the research aim and is also OO specific, including the accuracy to which OO can be observed. These interrelations show the complexity of the dependence between $OO, CO_I, CO_{II}, CO_{III}$, models of OO, CO_I, the aims of modeling, problems related to modeling and the principles of modeling.

The aims of model building can be as follows:

- reflecting, imitating or creating an image of the original or copy objects OO, CO_I,
- verifying or falsifying a theory,
- measuring the states of phenomena, processes and systems, both real and virtual,
- interpreting and explaining empirical, experimental measurements of the states of processes, systems and other research or real life objects,
- discovering regularities in the behavior of observed processes and systems,

- discovering critical (extremal) behavior of systems and processes,
- recognizing the causes of observed states, as well as regularities and irregularities in processes or systems,
- recognizing causal relationships between the elements of OO, CO_I, as well as between states of processes, systems, nets of events or bundles of processes,
- discovering new technologies, technical goods, tools, consumption and investment goods,
- discovering new uses for known goods or tools,
- deepening the analysis or creating a synthesis of known theories in a given field,
- initiating, carrying out complex modeling studies via simulation, i.e. multiple use of a model in its initial form or an adapted form,
- using models to forecast the states of processes or systems, or the form of their dynamics,
- using models in decision making processes.

These aims leads to various models and the use of such models leads to different problems regarding modeling and principles of modeling.

Among the problems regarding modeling, we may distinguish the following:

- which models preserve the relative durability of the form of OO or CO_I?
- which models preserve the minimal number of important features of OO or CO_I, apart from shape?
- which models preserve the maximal number of important features of OO or CO_I, apart from shape?
- which models preserve the minimal number of important features of OO or CO_I, including shape?
- which models preserve the maximal number of important features of OO or CO_I, including shape?
- how can we guarantee that a model has qualities such as:

- calibrativity,
- clarity,
- correspondability to OO, CO_I,
- durability in reflecting the forms of OO and CO_I,
- elasticity of reaction to changes in the form of OO and CO_I,
- homeomorphicity,
- isomorphicity,
- preservation of the shape, structure, proportions, substance, cardinality (of the elements) of OO and CO_I,
- reflexivity, representability, robustness,
- stability, especially structural stability,
- transparentability, translationability,
- unifiability,
- great conformity (closeness, consistency) to OO and CO_I,
- user friendliness in research, forecasting or decision making,
- ease of "taming" a model (becoming familiar with the model, computational simplicity of the model in itself and in applications, ability to gain a sophisticated understanding of the studied object)?

It is easily seen that it is impossible to formulate an ideal model that has all of these qualities. We will be lucky if we can build a model which possesses a few of them.

There are tacit and subtle relationships between the aims of model building mentioned above, the quality of the model built, problems regarding modeling and the principles of modeling.

Among the aims of model building, we can distinguish:

a1) explaining causes, the functioning of systems and the relations between causes and effects of events, processes or phenomena (real or virtual):

a2) forecasting (predicting) future events, processes and phenomena, as well as the functioning of systems;

a3) simulations based on the models built, in order to gain a better understanding of the detailed properties of these models or to grasp real or quasi-real experimental knowledge about the qualities of models treated as pictures of the real or virtual world;

a4) decision-making.

The qualities possessed by models are connected to, if not specific, to these aims. What are these qualities? In common language, models can be described e.g. as very good, good, average, bad or very bad. When assessing a model, one should, however, take other qualities into account, such as:

q1) the degree to which the aims of model building are fulfilled (aim fulfillment);

q2) the degree to which a model can solve a given problem (potential for problem solving);

q3) the degree to which the principles of modeling are fulfilled by the model constructed (fulfillment of the principles of modeling);

q4) the robustness of a model to changes in the elements or structure of a model;

q5) the level of stability and elasticity of a model with respect to operations or their elements and structures;

q6) the degree to which a model corresponds to the objects modeled;

q7) the level of simplicity of a model in its form and use;

q8) the degree to which a model covers the appropriate domain of scientific theory;

q9) the level of originality;

q10) the level of practicality.

The aims and qualities possessed by a model depend on the problem being modeled. Such problems are generated by particular practical problems, cultural needs, the needs of scientific research, production needs, the need for political power. Examples of such problems (questions) are:

t1) how do cancer cells come into existence?

t2) are non-equilibrium states of physical systems the most significant causes of changes in the forms and structures of Nature?

t3) how can the set of critical values of state parameters be practically adapted and used in order to optimize chemical processes?

t4) how should a space rocket be designed in order to fly to Venus?

t5) how should robots be designed in order to pilot space-vehicles within and outside the solar system?

t6) how should economic policy be designed in order to steer an economy and avoid economic crises?

t7) how should election processes and the political system be designed in order to be fair with respect to all social groups and enable continuous growth in the economic and social sphere while minimizing disturbances?

From the above discussion, the aims of modeling, principles of modeling, problems related to modeling and qualities (characteristics) of models have tacit connections.

There is probably no good set of principles for modeling which is consistent with all of these aims, problems and qualities. It is, however, possible to find good, if not the best, principles for modeling according to the problem being modeled and the aims of modeling.

What are the principles of modeling? As mentioned above, the principles of modeling depend on the subject of the model, the aims of modeling, the type of model used and its characteristics.

The general principles of modeling are as follows:

p1) to seek universal truths of Nature,

p2) conformity of a model with a given theory,

p3) conformity of the principles of creating (selecting) a model with the principles of formulating a theory,

p4) conformity of model selection with empirical data,

p5) conformity of model selection and studies with a data sampling generator (DSG),

p6) conformity of the principles of model selection with the principles of selecting a DSG,

p7) conformity of selected principles of simulation (for a given model) with the principles of modeling,

p8) conformity of the goals of simulation with the goals of modeling,

p9) keep modeling until your model is a homeomorphism and as close to an isomorphism as possible,

p10) when modeling ensure that the simulars (copies) are close to the simulanda (original objects) while the simular remains as simple as possible,

p11) when modeling preserve such properties as: equivalence of the model to the OO (original object), similarity of the model to the OO, robustness of the model (reflexivity, symmetry without transitivity, partial ordering),

p12) keep, if possible, Leibnitz's principium identitatis indiscernibilium, and, at least partially, the identization principle (i.e. the principle of preserving equivalence, identity, conformity, isoformity, non-extinctiveness),

p13) when modeling try to preserve the heritage of information and ensure computability of the model,

p14) ensure that it is easy to understand models and interpret them, together with their results,

p15) be cautious when evaluating the results obtained from modeling and adapting a model,

p16) try to select such models that achieve the chosen aims of modeling and properties of the model, while reflecting the nature of the subject being modeled.

p17) be creative in model selection and verification and search continuously for an ultimate theory and its model.

These principles can only be fulfilled to a certain degree, both according to the number of principles satisfied and the degree to which each is satisfied. We should, however, attempt to fulfill them to the highest possible degree.

3.5. Final remarks

The content of this chapter makes a formal basis for the chapters 4–10. The included definitions of Homm, Isomm, M, T may have more particular forms in concrete empirical applications. Practitioners can use a scheme of model aims, qualities, principles of modeling given in Section 3.4 that are more extensively discussed in the next Chapter 4.

Bibliography

Archangelski, W., Ponomarev, W. (1974). *Osnowy obstchey topologii w zadatchah i upraznenyah.* Nauka, Moskva.

Ashby, W. R. (1958). *An introduction to cybernetics.* Chapman and Hall, London.

Barwise, J. ed. (1977). *Handbook of mathematical logic.* North Holland, Amsterdam.

Borkowski, L. (1977). *Logika formalna.* PWN, Warszawa.

Bourbaki, N. (1964). *Éléments de Mathématique, Algébre II, III.* Hermann, Paris.

Bronschtein, N. et. al. (2009). *Nowoczesne kompendium matematyki.* Wydawnictwo Naukowe PWN, Warszawa.

Bulinski, A., Shiryaev, A. (2003). *Teorya slucaynyh processov.* Fismatlit, Moskva.

Császár, A. (1978). *General topology.* Akademiai Kiado, Budapeszt.

Dieudonné, J. (1968). *Algébre lineaire et géometrie elementaire*, Paris.

Edwards, R. (1965). *Functional analysis.* Holt, Rinehart & Winston, N.Y.

Engelking, R. (1976). *Topologia ogólna.* BM–47, PWN, Warszawa.

Frey, G. (1969). *The use of concepts "Isomorphic" and "Homeomorphic" in epistemology and the theory of science.* Ratio Vol. 11, No 1.

Gastev, Y. (2009). *Gomomorfismy i modeli.* URSS Moskva.

Grobler, A. (2006). *Metodologia nauk.* Aureus, Wyd. Znak, Kraków.

Grzegorczyk, A. (1969). *Zarys logiki matematycznej.* PWN, Warszawa.

Heller, M. (2006). *Filozofia i wszechświat.* Universitas, Kraków.

Hirzebruch, F. (1966). *Topological methods in algebraic geometry.* B131 Springer-Verlag N.Y.

Jänich, K. (1984). *Topology.* Springer-Verlag N.Y.

Kantorowitz, L., Akhilov, G. (1984). *Funkcjonalnyi analiz.* Nauka Moskva.

Keisler, H. J. (1971). *Model theory for infinitary languages.* North-Holland, Amsterdam.

Kelly, J. L. (1957). *General topology.* D. Van Nostraud Co, Princeton N.J.

Klein, F. (1872). *Vergleichende Betrachtungen über neuere geometrische Forschungen.* Deichert, Erlangen.

Kolmogorov, A. (1933). *Grundbegriffe der Wahrscheinlichkeitsrechnung.* B2H3 Springer Berlin.

Kosniowski, C. (1980). *A first course in algebraic topology.* CUP, Cambridge.

Kostrikin, A. (1977). *Vvedenye v algebru.* Nauka, Moskva.

Kuratowski, K. (1933). *Topologie I.* Warszawa.

Kutateladzhe, S. (1983). *Osnovy functionalnogo analiza.* Nauka, Nowosybirsk.

Lang, S. (1970). *Algebra.* Addison-Wesley, Reading Mass.

Malcev, A. (1970). *Algebraiceskiye sistemy.* Nauka Moskva.

Malcev, A. (1975). *Osnovy lineynoy algebry.* Nauka, Moskva.

Milo, W., et al. (2010). *Stabilność rynków kapitałowych a wzrost gospodarczy.* Wydawnictwo Naukowe PWN, Warszawa.

Möbius, A. (1863). *Theorie der elementaren Verwandtschaft.*

Naimark, M. A. (1955). *Normirovannye koltza.* Nauka, Moskva.

Naimark, M. A. (1976). *Teorya predstavlenyi grupp.* Nauka, Moskva.

Neveu, J. (1970). *Bases mathématiques du calcul des probabilities.* Masson, Paris.

Poincare, H. (1885). *Mémoire sur les courbes définiés par une équation differentielle.* J. de Math, Pures Appl.

Postnikov, M. (1979). *Lineynaya algebra i differentzyalnaya geometrya.* Nauka, Moskva.

Przeworska-Rolewicz, D. (1988). *Algebraic analysis.* D. Reidel Publ. Co., Dordrecht.

Robertson, A. P. (1964). *Topological vector spaces.* CUP, Cambridge.

Switzer, R. (1975). *Algebraic Topology.* Springer-Verlag, Berlin.

Tarski, A. (1936). O pojęciu wynikania logicznego. *Przegląd Filozoficzny*, T. 39.

Tarski, A. (1954). Contributions to the theory of models I, II. *Indagationes Mathematicae*, T. 16.

Tarski, A. (1955). Contributions to the theory of models III. *Indagationes Mathematicae*, T. 17.

Thom, R. (1972). *Stabilité structurelle et morhogénesé.* Benjamin, Reading Mass.

Waerden van der, B. L. (1967). *Algebra.* Springer, Berlin.

Woleński, J. (2005). *Epistemologia.* Wydawnictwo Naukowe PWN, Warszawa.

Chapter 4

On models and modeling

Summary

This chapter contains a brief, specific account of topics connected with conditions for constructing successful models and applying them successfully. It develops discussion given in Chapter 3. As it will be shown, the concepts 'model', and 'modeling' are very complex and capacious. They vary, sometimes significantly, in their meaning in different disciplines of science and the arts, as well as according to the type of human activity or profession. The basic thoughts, notions, ideas and concepts, used herein are well known from the English language literature on the subject. We hope that important new results, first published in other languages, have been absorbed by researchers using English as their medium for communication. Some of the thoughts and results presented here seem to be novel.

Keywords: model, kinds of models, modeling, principles of modeling

4.1. Introduction

There is a vast literature (in many languages) concerning models and modeling. Most articles, conference papers and books on the subject are devoted to problems related to modeling that are encountered in everyday life. These problems include the practicalities of constructing, testing and using models.

The meaning and form of a model varies according to the field of science or the arts, as well as the domain in which the model is applied. A model may take the form of a mathematical expression (e.g. an equation or system of equations), a graphical scheme representing the connections between the abstract or physical objects being studied, a picture or sculpture of a real object, or a micro-scale copy of a physical object, such as a car or ship. In science and the arts the objects being modeled can be real, fictional, mental or virtual. The following are types of object that can be modeled: abstract ideas and concepts, empirical or abstract theories, real systems and their behavior or changes in space and/or time, as well as virtual or artificial worlds.

This chapter will discuss certain important ways of understanding the concepts of model and modeling. In some cases, the author has reshaped the form and content of the definition of a model and the problem addressed by a model and discussed the consequences of these changes.

In Section 2, we present some results of our analysis of the essence of a model. Twelve fundamental questions concerning models are posed. A brief discussion of these questions is carried out both in this section and in the further text.

Section 3 contains a description of a categorization of models. This categorization is constructed according to four criteria.

Section 4 presents the results of a brief analysis of the factors that determine the variety of principles and methods of modeling. It should be noted that these factors are preceded causally by factors that influence human activities and goals in general, and the process of modeling and its goals in particular. This list of factors is certainly not exhaustive, but it is quite extensive with respect to the literature cited here.

Section 5 contains a description of some chosen problems of model building and modeling in economics. The problems in this field are very complex and only partially solvable. We refer to these ideas in applying A. Smith's theory of the causes of the wealth of nations – to describe economic growth in Poland in the transition period 1993–1998. This section also includes some comments on the specificity of constructing and using models in economics as a scientific discipline and in practical decision making within large corporations or central banks. The list of unsolved problems should be interesting to economists.

4.2. The essence of models

It is worthwhile to remind the reader that the essence of a thing means the most fundamental, important, significant, true, necessary, persistent and constant properties of the thing. This indicates, therefore, that these properties are not secondary, accidental, random or coincidental.

Note 4.1. It follows from the meaning of the word 'essence' that a modeler's aim should be to search for essential models.

It is astonishing why the notion 'theory' has been so popular in use from the time of the classical Greek philosophers, but the accompanying concepts of modeling and model were waiting to be invented for such a long time and only really developed in the XX c. Does the rapid progress in technology in the XX c. in itself give sufficient explanation of this fact, in particular considering the inventions of the radio, television, computer, Internet, photocopier, digital printers and cameras, tablets, smartphones etc.? Certainly, these modern tools have enabled the use of models and modeling to spread rapidly both lexically, as well as in making and realizing decisions. Another factor which partially explains the rapid spread of models is the enormous level of use of scientific tools (to which models and modeling also belong) in all domains of human activity in the XX and XXI c. A third factor determining the rising popularity of models and modeling is the growing power of media journalists, and internet users addictive use of the virtual internet world rather than the real physical world. As it will be seen later in the text, the complexity and usefulness of models and modeling are important constraints. This feature is particularly underlined by theory-oriented scientists.

Another feature of the literature on models is the use there of many kinds of models with increasing formal, semantical and semiotic complexity, in order to increase the impression of scientificity. In mathematics, mathematical logic and logic this feature is natural and necessary. The most advanced formal definitions of concepts (including the concept we are analzying here, i. e. model) were first proposed in mathematical logic and mathematics. Later, this extended to theoretical physics, logic and engineering. Quickly, the concept of model became indispensable in applied and theoretical physics, chemistry, astronomy, astrophysics, biology, economics, informatics, philosophy of science, medicine, biophysics, biochemistry, biometrics, chemometrics and econometrics. The language and methods of modeling have recently become very fashionable in psychology, sociology, political science and even in religious studies.

In order to structure further discussions and presentations, let us consider the following fundamental, essential questions:

4.2.1. Fundamental questions

q1. What are a model's aims?

q2. What are a model's objects?

q3. What is a model?

q4. What is a model's domain?

q5. What is a model's co-domain?

q6. What forms of model are there?

q7. What kinds of model are there?

q8. What are the means for realizing a model's aims?

q9. What are the criteria for evaluating a model?

q10. Does model selection depend more on the modeler's preferences and skills or on the user's preferences, needs and skills?

q11. Which methods or principles of modeling are necessary and which are sufficient for successful modelling?

q12. Which methods of modeling are most popular among economists and why this is so?

To keep the length of this text reasonable, we will give relatively brief answers to these twelve questions. Hopefully, the answers will be suitably instructive and concise, as well as deepen the ability of modelers to construct and apply models.

From a quick comparison of the questions in the set $\{q1,\ q2,\ \dots\ ,\ q12\}$, we can come to the following conclusions:

- questions 10 and 12 are linked directly either to modelers or the users of models,
- questions 1 to 9 deal with the notion of model either directly ($q3$, $q4$, $q5$, $q6$, $q7$, $q8$) or indirectly ($q1$, $q2$, $q9$),
- question $q11$ is linked to both modelers and the users of concrete models, as well as with the knowledge of modeling accumulated in the form of papers, books and manuals.

More detailed, though succinct in form, answers will follow under the appropriate labels *Ad q*1, *Ad q*2, ... , *Ad q*12.

Ad q1. The aims of a model will here be identified with the ways in which a model can be used. Thus models can be used in many roles.
q1*a*. Models as tools or devices for recognizing:

1) physical things (natural or man-made) features,

2) words or expressions of natural or scientific language of particular professions, arts, artificial languages, the importance of language in the interpretation of a model's findings, including natural, scientific and artificial language, the terminology of particular professions and the arts

3) the suitability of concepts, ideas and mental worlds presented in novels, poetry, paintings, sculpture, films, songs, music, video-clips and video-games;

4) the validity of the rules, principles and methods of modeling inherent in a model's form and content;

5) the validity of theories, truthness, falsity, consistency, coherence, decidability, completeness, decisiveness, verifiability, describability, cognitivity, predictability, informativeness, usability in terms of solving real problems, closeness to reality, reach of description and explanation, flexibility with respect to anomalous changes in the objects of the theory, robustness of conclusions and predictions.

Note 4.2. The last two items are especially important in research.

q1*b*. Models as aids (tools) to improve the quality of:

1) the understanding and observability of modeled objects, their description, prediction, as well as the potential for decision making,

2) the results of modeling and theorizing,

3) recognition of the properties of models and theories,

4) recognition of domains and co-domains of models and theories,

5) recognition of statistical patterns in real and experimental data,

6) the level of precision of scientific and non-scientific discussions about events, states, phenomena and processes in Nature, as well as about abstract concepts and ideas, models, theories, including possible, virtual and mental worlds,

7) the precision of words and expressions used in natural or scientific languages,

8) popular and scientific knowledge about the causes of atypical actual events, states and phenomena of Nature, the economies of particular countries or geo-politics,

9) conducting experiments that increase our knowledge about the physical or conceptual world,

10) exploration of recorded data aimed at detecting trends, cycles, irregularities, atypicalities,

11) applied methods of modeling, prediction ex ante, decision-making, control,

12) argumentation about the need to improve education in metrology, and the need of clear communication between data providers and data users (analyzers, describers, explainers, prediction-makers, decision makers).

Note 4.3. Points 1, 2, 3, 4, 6, 7, 8, 9 and 11 are especially important to theoretically oriented researchers. Empirically oriented researchers should be more concerned with points 5, 10 and 12.

q1*c*. Models as:

1) links, joints, tiers, intermediaries, mediators, connectors, bridges and associators between the objects modeled and observational data, modelers, users of models, assessors of models, traders of models.

Note 4.4. In these roles, a model aims at: linking, connecting, joining the objects to be modeled – where the quality depends on a modeler's skills – either to each other or connecting these objects to modelers, robots, sensors or simulators (which consist of computers, programing packages, the goals of simulation). When the objects are persons in the role of subjects of research, models serve as intermediaries in the research process by enabling the formulation of the study's aims, or as a basis for evaluating and classifying the subjects or as a basis for the evaluation of data quality.

q1*d*. Models as:

1) detectors of changes,

2) verifiers of changes,

3) interpreters of changes.

Note 4.5. Changes may appear in the form and context of concepts, ideas, in the structure of the model or the objects modeled, or in the conditions regarding the space-time invariance of the objects modeled, as well as in the scope of the skills of modelers to observe and predict and the skills of both modelers and users of the model to evaluate, together with their habits and ideology.

q1*e*. Models as:

1) displays of modelers' motives, aims, abilities and skills,

2) descriptions of objects' roles, essence and environment,

3) explanations of causes and effects of changes, as well as of objects' status and evolution in time,

4) predictors of changes in objects in time, space or form,

5) basis for fantasy games,

6) tools for decision making.

Note 4.6. The first role is recognized and stressed by psychologists and artists. The second, third and fourth roles are particularly respected by researchers and educators in the formal natural sciences, as well as economics, management and sociology. The fifth and sixth roles of models are preferred by practitioners in many non-scientific professions.

q1*f*. Models as:

1) hybridization of several models or their parts, and coupling data to suitable theories,

2) *ad hoc* fits to anomalous data.

Note 4.7. In both of these roles, models aim to unite, align, order and transform such classes of entities as data, known opinions, convictions, common and scientific knowledge, and research habits. By virtue of the complexity of social and economic systems these roles are particularly well suited to research in sociology, social psychology, political economics, economic history or the behavior of political parties.

q1*g*. Models as:

1) ingredients of the basis of investigations into artificial intelligence, basis of inventions,

2) prototypes of robots, sensors or machines.

Note 4.8. Such models are constructed and used in engineering, informatics, modern transportation and telecommunication systems, including cosmonautics.

q1*h*. Models as:

1) devices to ease the reasoning process, write papers, monographies, textbooks, manuals, and even commercial advertisements or practical guides/expert assessments.

2) source of creative activity and enjoyment for modelers.

Note 4.9. Both of these roles correspond to the activities of either modelers or experts who know the properties of these models.

In the above discussion of the aims of modeling, we either directly or indirectly referred to the objects of models. We may classify the objects of models as follows (the symbol denoting each one is given in brackets):

Ad q2.

q2*a*. objects of a physical-chemical nature ($\mathrm{Ob}^{\mathrm{phc}}$-s),

q2*b*. objects of a biological nature ($\mathrm{Ob}^{\mathrm{bio}}$-s),

q2*c*. human beings ($\mathrm{Ob}^{\mathrm{hb}}$-s),

q2*d*. objects from the fields of physics and chemistry ($\mathrm{Ob}^{\mathrm{PhC}}$-s),

q2*e*. objects from the field of biology ($\mathrm{Ob}^{\mathrm{Bio}}$-s),

q2f. objects from the social sciences (Ob^{SS}-s),

q2g. objects from linguistics and informatics (Ob^{LI}-s),

q2h. objects from the fields of mathematics, mathematical logic and logic (Ob^{MMLL}-s),

q2i. objects from the fields of engineering and technology (Ob^{ET}-s),

q2j. objects from the fields of economics and management (Ob^{EM}-s),

q2k. objects from the field of medicine (Ob^{M}-s),

q2l. artistic objects (Ob^{Art}-s).

Other classifications of objects are possible. They may be more detailed or less scientifically oriented. In general, objects are either real and physical (see the categories denoted by $q2a$, $q2b$, $q2c$), or abstract, artificial and mental (such as $q2g$, $q2h$, $q2l$), although they can have features of both of these traits. However, it should be stressed that within the first triad of categories the set of common designates sharing common physical-chemical elements or particles is not empty. The same can be said about the set of abstract concepts with common semantic content.

Using the symbols denoting these categories of objects, we may formulate the following remarks: Each Ob_i^g, $i = \overline{1, m_i}$, $g \equiv phc,\ bio,\ hb$ contains (is built from) the same basic particles of matter. Every Ob_j^g, $j = \overline{1, m_j}$, $g \equiv LI,\ MML, Art$, includes, in turn, abstract, fictitious symbols in written form or as a real, physical sound form with a direct or indirect correspondence to all other groups of objects. Objects belonging to the groups $q2d$, $q2e$, $q2f$, $q2i$, $q2j$, $q2k$ are objects from the fields of particular sciences.

Thus, letting $q2p$, $p = a,\ b,\ \ldots\ ,\ l$, denote the set of objects in group p, we may assume that for many $p \neq r$, $p,\ r = a,\ b,\ \ldots,\ l$, the intersection $q2p \cap q2r$ is not the empty set $\emptyset$. Moreover, when $p,\ r = g$ and $g = d,\ e,\ f,\ i,\ j,\ k$, the intersections of sets $q2p \cap q2g$ and $q2r \cap q2g$ are not empty. The same can be said about the intersection of sets $q2h \cap q2h^{'}$, where $h, h^{'} = d, e, f, i, j, k$, due to the diffusion of methods and modeling into interdisciplinary science and fields of knowledge, as well as fashionable approaches to research topics, methods, models and theories.

It is also worthwhile to classify the objects of models according to criteria regarding the general methodology of science or concepts characterizing human activities and their products. Hence, using such criteria, we may categorize the objects of models as follows:

Q2a. theories of systems (i. e. real, abstract, fictional or virtual systems),

Q2b. real-actual phenomena and processes, fictional-virtual phenomena and processes,

Q2c. theories of empirical or abstract processes,

Q2d. benchmark models or theories,

Q2e. real bodies and systems,

Q2f. real life plans, projects and designs, programs of a technical, economic or socio-political nature,

Q2g. solving tasks and problems: systems of equations, games, conflict, forming a system for cooperation or competition, realizing an investment program, numerical solutions, forming a sculpture or small scale prototype,

Q2h. motion in transportation systems (buses, trains and airplanes), production systems, functioning of a company or corporation,

Q2i. legal acts, rules, codes, legal systems, socio-political systems, safety systems (internal and external), economic-financial systems, healthcare systems, education and research systems,

Q2j. processes of measurement, prediction, simulation, decision making and realization, capital accumulation, goods production, division of goods, exchange and circulation of goods, market segmentation,

Q2k. rationality and efficiency of decisions: political and economic decisions at the level of local and central government, assigning workers to positions within an organization, dividing tasks among staff and monitoring the degree to which tasks are realized.

Note 4.10. According to this classification, objects belonging to the groups $Q2a$, $Q2c$, $Q2d$ are abstract in form, although in content they also deal with reality. The objects in groups $Q2b$, $Q2e - Q2k$ are, in turn, strongly related to the real life problems of individuals, workers, owners of capital, politicians, those ensuring security and all types of managers.

The answer to the following question: "What is a model?" is particularly important to modelers, as well as the users and assessors of models.

Ad q3. The answer to this fundamental question is connected with interpreting the essence of the concept of a model. Such an interpretation is carried out from the point of view of the form of a model and the semantic content contained in its designates and predicates. Based on the literature cited at the end of this chapter, models may be categorized into the six following groups:

q3*a*. A formal set, structure or ideal acting as a concrete or general example, reference, summarizer, verifier, referee of the:

1) validity, suitability, truthfulness of axioms, assumptions, sentences, rules of reasoning, rules of modeling, rules for evaluating a model, rules for evaluating the consequences of the assumed structures and ideals on the quality of the description of the abstract and real objects modeled;

q3*b*. A map, mapping, transformation, transformate, homomorphism, isomorphism, correspondence, operator, relational system, relation or function;

Note 4.11. The interpretations of a model given in *q*3*a* and *q*3*b* are formal. Their formality originates from the form and content of the terms used, which have fixed meanings in mathematics and mathematical logic. These terms are also very popular and have similar meanings in physics, engineering, biometrics, econometrics, chemometrics and informatics. The growing importance of interdisciplinary research, internationalization of research programs and the ease of access to old and new results of studies (both theoretical and practical, though not all free of charge) thanks to the Internet have made it possible to speed up the diffusion of the concepts given in *q*3*a* and *q*3*b* to such disciplines of science as: chemistry, biology, economics, management, operations research and games, and even medicine, sociology, psychology, the philosophy of science and history.

q3*c*. One or a collection of objects acting as a representation of objects which can be modeled, of transformed objects, the essence of Nature, human constructions, sets of utterances, propositions or sentences being a part of a theory;

q3*d*. A well structured source or repository of empirical information (i.e. measurement data, facts about natural phenomena, socio-demographic or economic data, filtered empirical data),

q3*e*. A model based simulator (containing a model, computer program, Internet or network application) that enables data grouping, data processing, data mining, data analysis, data interpolation or extrapolation, data approximation, simulations, the graphical representation of data and interpretation of data (statistical data and data from simulations);

q3f. A representer (representation) or realization of a theory; copy of a theory; artifact of a theory or the structure of a real or abstract object; invariant of a theory, bearer (carrier) of a theory's meaning, representation, validation, truthness, falsity or causality; a theory's truth-maker, referee, interpreter or refuter.

Note 4.12. It is worthwhile to recall that we use the following interpretation of the word "essence". By the essence of a word, concept, notion, idea, sentence, model or theory, we understand their semantic and logical content and form that contain the most important, necessary, highly preserved properties of the objects represented by these expressions. The interpretations $q3c - q3f$ state that the essence of a model is reflected in its very significant practical features connected with modeling, theorizing, data generation, model-based simulation, model testing and searching for the causes of events and processes.

We treat these classifications of models' aims and objects, together with the brief remarks about the essence of models, as important, if not indispensable, ingredients of knowledge about models and modeling. This knowledge may be extended by suitable use of the concepts of a model's domains and co-domains.

4.3. Domains and co-domains of models

In further discussions, it would be practical to use symbols to denote certain important structural elements of models. To begin, let
$D \equiv D_M \equiv D(M) \equiv Dom(M)$ denote the domain of model M,
$CD \equiv CD_M \equiv CD\,(M) \equiv Co - Dom(M)$ denote the co-domain of model M,
W denote the world (set, collection, complex, universe of things),
W_m denote the set of material (real, physical) objects existing at a given time and place,
W_{nm} denote the set of non-material (non-physical) objects,
$\emptyset$ denote the empty set, $\cap$ the intersection and $\cup$ the union of sets,
$W \equiv W_m \cup W_{nm}$, and $W \neq \emptyset$, $W_m \neq \emptyset$, $W_{nm} \neq \emptyset$, $W_m \cap W_{nm} = \emptyset$.

We further split the set W_m into the sets W_m^{lb}, W_m^{nlb} of physically existing (at a given time and place) living beings (lb), and non-living beings (nlb). Obviously, $W_m = W_m^{lb} \cup W_m^{nlb}$, $W_m^{lb} \neq \emptyset$, $W_m^{nlb} \neq \emptyset$, $W_m^{lb} \cap W_m^{nlb} = \emptyset$. The set W_m^{lb} includes really existing (living): human beings, other animals (living on the Earth or in the sea), insects, bacteria, 1-cell beings and viruses.

The set W_m^{nlb} contains physically (really, factually) existing non-living beings such as: metals, fluids, other chemical solutions, atoms, particles, physical goods produced by humans, large bodies of matter (such as comets, planets, stars and stellar systems). The set W_{nm} covers groups of elements that are treated as non-material (non-physical). These include such objects as: words, concepts, ideas, propositions, sentences, connectives and other entities of natural or artificial languages formed by humans. The following are more advanced objects of the set W_{nm}: abstract lemmas, theorems in mathematics and logic, abstract models and theories, collated knowledge stored in the form of papers, articles, textbooks and monographies. The following also belong to W_{nm}: poems, novels, paintings, sculptures, songs, works of classical music (symphonies, operas), ballet or theater performances.

Remark 4.1. The inclusion of certain objects in W_{nm} is discussable. The objects of, e.g., poems and novels may be fictitious heroes and characters that do not exist physically or materially in the W_m-world. However, characters may be based on persons factually existing in W_m. Although physically performed (printed, painted, sculptured, sung, played) pieces of art are engraved in a physical matter, they have immaterial conceptual content from the point of view of modeling and theorizing. With some reservation, one can accept the view that the content of objects contained in W_{nm} is recorded (memorized) in the physically existing brain of a concrete person belonging to W_m^{hb} and takes the physical form of occupied brain cells. In this case, abstract notions in the non-material world have a representation (copy, counterpart) in a physical form. Hence, objects of art existing in a physical form belong to W_m and thus one can accept that $W_m \cap W_{nm}^{Art,\ ph.} \neq \emptyset$.

It should be stressed that there is no unique way of understanding the concept of the domain of a model. The clearest interpretation can be found in mathematics and mathematical logic. If a model is of the form

1. $M \equiv (y = f\,(\mathrm{x})\,,\ y \in \mathbb{R}, \mathrm{x} \in \mathbb{R}^k, f : \mathbb{R}^k \to \mathbb{R})$,

then mathematicians say that

$D\,(M) \stackrel{\text{def}}{=} \mathbb{R}^k$ or $D(M) \stackrel{\text{def}}{=} \left\{\mathrm{x} \in \mathbb{R}^k\right\}$.

The co-domain is defined as

$CD\,(M) \stackrel{\text{def}}{=} \mathbb{R}$ or $CD(M) \stackrel{\text{def}}{=} \{\mathrm{y} \in \mathbb{R}\}$.

The suitability of f means that application of f guarantees the existence of $y \in \mathbb{R}$. This definition of a model suggests the following simple postulate:

2. $M\,:\,\Phi(D\,(M)) \to \Phi(CD\,(M))$, or $\Phi(D\,(M)) \to \Psi(CD\,(M))$, $\Phi = \Psi$ or $\Phi \neq \Psi$,

which holds for all models, where $\Phi(D\,(M))$ is the set of the features of objects chosen to define the domain, and $\Phi(CD\,(M))$ or $\Psi(CD\,(M))$ is the set of features of these objects to be explained (modeled) by the features $\Phi(D\,(M))$.

Another, somewhat unclear, case occurs when we link models to theories. Suppose that we define a theory $T \stackrel{\text{def}}{=} \{$*internally connective theorems or laws*$\}$. The domain of the theory T can be defined as follows:

3. $D(T) \stackrel{\text{def}}{=}$ {the elements (objects) of the system S described by the theory T, where S is the set of objects in the system, together with the time period covering the existence or investigation of S and the features $P_1, \ldots, P_m$ of the objects}.

More attractive, especially to the users of models, is the general, highly intuitive and qualitative statement that:

4. $D\,(M) \stackrel{\text{def}}{=}$ the non-empty set of values or features of all the entities (arguments) of model M, where the arguments may be variables or parameters of the formal relations of M, or the logical values of sets of axioms, premises sentences, encountered in M.

Definition 4 is free from any dependence on T and assumptions regarding the spaces of the values of the arguments.

How should $CD(T)$ and $CD(M)$ be defined according to Definitions 3 and 4? The follow are useful definitions of CD:

3a. $CD(T) \stackrel{\text{def}}{=}$ the set of utterances, propositions, sentences, theorems and forecasts of the future features $\hat{p}_1, \ldots, \hat{p}_m$ of S-objects;

3b. $CD(M) \stackrel{\text{def}}{=}$ the set $Cn(\mathbb{A}_M)$ of conclusions resulting from the axioms of the model M, $\mathbb{A}_M$, and empirical data.

More precisely, as a background, we now use the definitional scheme for a mathematical theory as represented by Expression 3 in Section 3 of Chapter 10 in this text ("On theories and theorizing"). Using this expression, we may formulate the following structure for defining mathematical models:

5.
$$\mathbb{M}\,(\mathbb{T}^m) \equiv (\mathbb{L}_\nu^{m,\ \mathbb{M}}, \mathbb{L}^{m,\ \mathbb{M}},\ \mathbb{L}_o^{m,\ \mathbb{M}}, \mathrm{I\!R}^{m,\ \mathbb{M}}, \mathbb{DEF}^{m,\ \mathbb{M}}, \mathbb{A}^{m,\ \mathbb{M}},$$
$$\mathbb{C}\mathrm{n}(\mathbb{A}^{m,\ \mathbb{M}}),\ \Theta^{m,\ \mathbb{M}}\hat{\alpha}\mathbb{P}^{m,\ \mathbb{M}}, \mathbb{EC}^{m,\mathbb{M}}).$$

It would be practical to assume that:

$$\mathbb{L}_\nu^{m,\ \mathbb{M}} \subset \mathbb{L}_\nu^{m,}, \mathbb{L}^{m,\ \mathbb{M}} \subset \mathbb{L}^m, \mathbb{L}_0^{m,\ \mathbb{M}} \subset \mathbb{L}_0^{m,}, \mathbb{IR}^{m,\ \mathbb{M}} \subseteq \mathbb{IR}^m, \mathbb{DEF}^{m,\ \mathbb{M}} \neq \mathbb{DEF}^m,$$

$$\mathbb{A}^{m,\ \mathbb{M}} \neq \mathbb{A}^m, \mathrm{Cn}(\mathbb{A}^{m,\ \mathbb{M}}) \neq \mathrm{Cn}(\mathbb{A}^{m,}), \Theta^{m,\ \mathbb{M}}\hat{\alpha}\mathbb{P}^{m,\ \mathbb{M}} \neq \Theta^{m,}\hat{\alpha}\mathbb{P}^m,$$

$$\mathbb{EC}^{m,\mathbb{M}} \neq \mathbb{EC}^m.$$

In the above statements, the bold letter $\mathbb{M}$ written as a superscript denotes a model or family of models. The other symbols have the same meaning as in the cited text. The relation between the sets given above signifies that the languages used by models are narrower (less rich) in words, connectives and semantic content. The list of inference rules used for models is at most as numerous as the list of such rules for theories. The corresponding lists of definitions, axioms, their consequences, theorems, proofs and evaluation criteria for models and theories are assumed to be different.

Based on Definition 5, we define

5a. $D(\mathbb{M}(\mathbb{T}^m)) \overset{\text{def}}{=} \mathbb{T}^m$

5b. $CD(\mathbb{M}(\mathbb{T}^m)) \overset{\text{def}}{=} (\mathbb{DEF}^{m,\ \mathbb{M}}, \mathrm{Cn}(\mathbb{A}^{m,\ \mathbb{M}}), \Theta^{m,\ \mathbb{M}}\hat{\alpha}\mathbb{P}^{m,\ \mathbb{M}})$.

Hence, novelties in definitions, the consequences of the axioms used in $\mathbb{M}$, as well as new or modified models and theorems are taken to be the CD of $\mathbb{M}(\mathbb{T}^m)$.

It would be epistemically interesting to trace which aims, objects, essential qualities, domains and co-domains of models are most associated with the groups of models discussed below.

4.4. Kinds of models

There are many criteria for classifying models that can be found in the literature given at the end of this chapter. The first classification to be considered is based on the ontic status of the modeled objects, that is to say the matter out of which these objects were or are formed or built. According to this criterion, we may classify models into the following groups.
(ck1): CD(M) oriented classification

a) models of solid form bodies composed of physical-chemical matter,

b) models of fluids (gas, water),

c) models of biological beings,

d) models of human beings,

e) models of the means of human communication, including:

- models of natural languages,
- models of artificial languages,
- models of the languages of scientific disciplines,
- models of artistic languages,
- models of professional languages,

f) models of the production of physical goods, products created by human beings,

g) models of the human mind, ideas and concepts,

h) models of previously or now presently existing phenomena, processes and states,

i) models of fictional, virtual phenomena and processes.

It is easily seen that modeled objects $\hat{W}_m$ from the groups a, b, c, d, and f belong to the previously defined set $W_m \equiv W_m^{lb} \cup W_m^{nlb}$. Modeled objects $\hat{W}_{nm}^{(h)}$ from the group h are characterized by physical or environmental processes in space and/or time: processes describing particular individuals or groups of individual living beings or describing non-living material objects. Analyses of these processes provide research material for formulating suitably justified properties of the objects belonging to W_m. Modelers who do not believe in the possibility of finding the factual causes of the states the objects studied, or of successfully inferring causes from observed events, phenomena and processes, treat only the group of models (h) as acceptable research tools. Thus, since they do not strive to discover the essence of the things studied, they are non-essentialists (phenomenalists).

The second class of models described in (ck1) includes groups e, g and i. The modeled objects $\hat{W}_{nm}^{(e,g,i)}$ belong to the set W_{nm} and include components of languages of any type, abstract concepts and ideas, as well as abstract mental phenomena or processes. Some of them, e.g., objects from science-fiction may come to exist as real objects and thus become members of the set W_m or $\hat{W}_m$.

The second criterion for classifying models is based on academic discipline. In this way, we may define the following classes of models.

(ck2): kind of scientific discipline oriented classification

a) models of objects analyzed in physics and chemistry,

b) models of objects analyzed in biology, biophysics and biochemistry,

c) models of objects analyzed in medicine and psychology,

d) models of objects analyzed in mathematics,

e) models of objects analyzed in mathematical logic and logic,

f) models of objects analyzed in engineering and informatics,

g) models of objects analyzed in econometrics, sociometrics and psychometrics,

h) models of objects analyzed in economics, sociology, psychology and political science,

i) models of objects analyzed in philosophy, history, arts and religion,

j) models of objects analyzed in pedagogy.

In classical and medieval times, science was practiced either by relatively rich persons or talented persons from the low income class selected by them, mostly in Europe. The Renaissance, Enlightenment and French Revolution speeded up the progress and division of science into many disciplines, especially into the so called natural sciences. This process of division was particularly spectacular in the XX c. There are further subdivisions of the disciplines considered above. These are sometimes too detailed and possibly restrain the accumulation of knowledge, due to a small number of researchers trying to synthesize the detailed results of investigations. These phenomena result from the great pressure of market forces (including mass education at the undergraduate and postgraduate level in many of the world's countries), competition between big corporations (including multinationals), governmental and parliamentary decisions about the financing of science and education. To illustrate these tendencies to subdivide, we present the following divisions of the subject of economics: microeconomics and macroeconomics; descriptive and mathematical economics, econometrics; theoretical and applied economics; managerial and financial economics. Finally, it can be easily observed that finance, by definition belonging to economics from classical times, due to the growing power of the banking system in the economy, is treated by financialists as a separate discipline. The same tendencies of subdividing can be seen on the part of people dealing with accounting and the organization of economic units. Summing up, we stress that based on this classification by discipline, there is recognizable interdisciplinary overlap between the groups a-g, f-h and i-j. Examples of models which combine knowledge from other combinations of disciplines can also be easily found.

The third criterion according to which we classify models is based on the type of theories which the models represent (via various means of research) or the type of reasoning used in the formulation of a theory and the features of a theory. According to this criterion, we can distinguish between the following types of model:

(ck3): type of modeled theory classification

a) inductive, deductive, reductive and abductive models,

b) descriptive, explicative and predictive models,

c) intuitive, sublimative, advanced and formal models,

d) static and dynamic models,

e) true, false and possible models,

f) epistemic, doxastic, scientific and non-scientific models,

g) holistic and partial models

h) categoric, modal, deontic, fuzzy and possibilistic models,

i) deterministic, probabilistic and stochastic models,

j) verifiable and non-verifiable models,

k) theoretical, applied and pragmatic models,

l) simple and complex models,

m) consistent, complete, coherential and decidable models,

n) robust, non-robust, stable and unstable models,

o) old and new models,

p) exact and approximative models,

q) cognitive, epistemic and operational models,

r) causality, non-causality, dependency and independency models,

s) factual (actual), artificial, fictional and virtual models,

t) typical, atypical and fetish models,

u) valid, proper, improper, invalid, suitable and justifiable models,

v) optimal control, optimality, non-optimality and almost optimality models,

w) rational and irrational models,

x) models describing experiments, models defining simulation procedures.

The classifications a, c, e, f, p, q, r, w, u and x are based on the methods of reasoning used in the formation of theories. The classifications b, g, l and x are based on the aims of models (and theories). Models of the types d, j, k, i, m, n, o, s and t are based on features related to how theories and their models are analyzed and evaluated. Models of the type g are based on the scope of a model and generality of a theorem.

Theoreticians will use conventions for constructing models related to the classifications a, e, f, g, h, i, j, k, m, n, p, r, w, u. Practitioners use conventions based on the remaining categorizations.

The last, fourth, criterion for classifying models used here is the defined aim when constructing a model

(ck4): aim of model classification

a) general use models,

b) specialist use models,

c) practical use models,

d) permanent use models,

e) temporary use models,

f) models used for exploration,

g) analytic models,

h) models of pattern recognition,

i) models used for experimental studies,

j) models for forecasting,

k) models for decision making and realization,

l) models aimed at extending cognizance, knowledge and skills,

m) models aimed at improving individuals' skills in modeling and theorizing

n) models creating fun and enjoyment.

This classification is not exhaustive. A more complete one should take into account the list of possible aims presented earlier. The one presented above only briefly underlines a few features of the uses of models that were not so highlighted by the previous list of aims.

4.5. More specific forms of models

The previous sections of the text referred to possible forms of constructed or formulated models. We start with analytic mathematical models. The most commonly applied objects in this discipline are functions. Therefore, there are as many models as there are types of functions. We may thus, f.ex. discern:
Analytical mathematical models

- model of a continuous, absolutely continuous function,
- model of an analytic function
- model of an algebraic function,
- model of an integrable function,
- model of a cyclometric function,
- model of an elliptic function,
- model of a wave function,
- model of a Green function,
- model of a Hamilton function,
- model of a Lagrange function,
- model of a Laplace function,

- model of a Riemann function,
- model of a complex function.

There are many more types of functions whose models are analytical in form. When we want to underline the geometric shape of a function's image, then the simplest division is the one distinguishing between linear and nonlinear models. Besides these forms, in other disciplines, the following forms of models are considered:

Other specific forms of models

- models expressed in logical form,
- models expressed in the form of drawings (figures, pictures),
- models expressed in the form of symbolic diagrams,
- models in the form of paintings,
- models in the form of sculptures (in stone, metal),
- models in the form of metallic constructions,
- models in the form of video disk recordings,
- models in the form of prototype consumer or investor goods,
- models in the form of genetically modified plants,
- models in the form of genetically modified animals,
- models in the form of genetically modified bacteria, viruses.

It is not difficult to observe that mass use of the objects obtained experimentally based on the last five groups of models (which belong to the set of objects W_m^{nlb} and W_m^{lb}) has brought unexpected (sometimes harmful) results in people's lives. It has also worsened the economic conditions of some people in less developed economies.

The classifications of models described above enable us to draw the following conclusions:

- sets of modeled objects $\{Ob\text{-s}\}$ are the sets $W = W_m \cup W_{nm}$ or the features, properties of W-elements, W-structures, or W-sub-structures,
- there exists $M : W \to \hat{W}$, where $\hat{W}$ are the properties of a model in the case of formal models, formal relations in M, or physical copies of original objects from W,
- there exists a correspondence (mapping) $M : D(M) \to CD(M)$,
- it often happens in modeling that we treat some elements of W_m or W_{nm} as models of other elements taken from W_m or W_{nm}. This is not the case when elements of $\hat{W}$ are treated as properties of elements of W.

In the next section, we will present a discussion of the problem of searching for optimal or near optimal models.

4.6. Modeling and principles of modeling

In further discussions, we understand "modeling" as being a process carried out by humans which satisfies certain conditions. It would be instructive to present such processes in an algorithmic form. Thus, the process of modeling comprises the following groups of actions:
grups of modeling actions

a) gathering data or materials,

b) investigation, classification or reclassification of data or materials,

c) data pattern recognition, forming lists of objects to be potentially modeled, in practice in the role of determinants or effects of determinants, i.e. $D(M)$ or $CD(M)$,

d) forming an initial version M_0 of model M linking $D(M_0)$ to $CD(M_0)$, i.e. $D(M_0)\ M_0\ CD(M_0)$ or $M_0 : D(M_0) \to CD(M_0)$ or $CD(M_0) = M(D(M_0))$,

e) testing, by the use of specified evaluation criteria (EC), the empirical and/or theoretical validity of M_0,

f) temporary acceptation or rejection (refutation) of M_0 using the results from (e),

g) additional studies of the features of M_0 (by practical use, by simulation based on M_0),

h) updating data and materials, adapting the lists of W_m, W_{nm}, $D(M_0)$, $CD(M_0)$ and features $\Phi(D(M_0))$, $\Phi(CD(M_0))$ to this new information,

i) reconstruction or respecification of M_0 to M_1, $D(M_0) \rightarrow D(M_1)$, $CD(M_0) \rightarrow \mathrm{C}D(M_1)$, $\Phi(D(M_0)) \rightarrow \Phi(D(M_1))$, $\Phi(CD(M_0)) \rightarrow \Phi(\mathrm{C}D(M_1))$,

j) testing the validity of M_1 according to the EC, and checking the robustness of M_1 by simulation,

k) deciding whether to accept or to refute M_1,

and further we need to repeat actions as above until some criterion for stopping is fulfilled.

Note 4.13. In the above scheme of actions, we omitted the dimensions of time and space. Strictly speaking, they should be reflected in the notation (e.g. $M_0(t_0)$, $D(M_0(t_0))$, $M_1(t_1)$, $D(M_1(t_1))$). If t_0, t_1 denote months or quarters, then the composition of $W_m(t_1)$ and $W_m(t_0)$ or $W(t_0)$ and $W(t_1)$ may change. It should also be added that the assumed structures of $D(M)$ and $CD(M)$, and the form of M vary from one modeler to another (even for fixed t).

In choosing a successful model M, the role of the modeler is crucial. Thus the success of a modeler depends on his abilities and experience in modeling, on his actual knowledge of the subject being modeled, on his skills, desires, needs, will and propensities, as well as on his professional and social status, together with the technical and financial means he possesses.

To some degree, everyone is modeling (forming) something. Their activities concern not only imitating observed objects, but also creating new objects, e.g. new physical objects, words, utterances, meals, social ties.

In general, the results of the process of modeling and the principles applied depend on the following factors (the lower subscript cc denotes a concomitant (within M) object):

factors of the quality of modeling

f1) the physical properties of studied objects belonging to W_m and $W_{m,cc}$,

f2) the physiological properties of studied objects belonging to W_m^{lb} and $W_{m,cc}^{lb}$,

f3) the physical structure of W_m-objects and $W_{m,\ cc}$-objects, f4) the degree of observability, cognizance and explicability of W_m-objects and $W_{m,cc}$-objects,

f5) the degree to which the essence of W_{nm}-objects, W_m-objects and $W_{m,cc}$-objects are understood,

f6) the degree of attainability of purposeful forms of physical objects in W_m and $W_{m,cc}$,

f7) the degree of transformability of W-objects and W_{cc}-objects,

f8) the degree of decomposability of W-objects and W_{cc}-objects,

f9) the degree of complexity of W-objects and W_{cc}-objects,

f10) the detectability of structure in single objects and in groups of them,

f11) the degree of analyticity of abstract theories,

f12) the magnitude or scale of W-objects and W_{cc}-objects,

f13) the relative share of qualitative and quantitative components in formal models and theories,

f14) the possibility of obtaining repeatable reconstructions of modeled objects and concomitant (in modeling, models) objects,

f15) the duration of an objects' existence,

f16) the intensity of the modeler's passion and endurance, as well as desire for perfection in modeling,

f17) his will and desire to learn new methods of modeling that use new technical means,

f18) his openness to new trends in modeling approaches,

f19) a modeler's keenness of understanding of an objects' features, roles, uses, mysteries,

f20) a modeler's endurance in the intensive, often highly repetitive, process of collecting sufficient data and checking their quality,

f21) his degree of respect to the principles of modeling,

f22) a modeler's abilities to convince owners of large capital to finance modeling projects, f23) a modeler's capacity to prepare good facilities and technical assistance for modeling,

f24) a modeler's engagement in promoting modeling and teaching how to model,

f25) creating positive opinions about the potential of modeling among influential decision makers.

According to the above classification, some of the factors are not controllable by modelers. These include factors f9, f12, f15. The other factors are more or less controllable. Factors f1-f3, f16-f20 refer to the effects of the personality and drive of a modeler which modulate the intensity and scale of the effort invested in modeling. The conditions for objectively observing research objects are reflected in factors f4-f8, f10-f11, f13-f14. The last five factors, f21-25, characterize the conditions for research, including the financial resources for research possessed by a modeler, and the research atmosphere existing in the appropriate country.

These factors determine, to a large extent, the results of the process of modeling, that is the cognitive, explicative, predictive and operational achievements obtained by using the resulting model. The role of the principles of modeling have not been discussed by us directly. Obviously, this role is either seen and used directly (according to specified principles) by many experienced modelers, or unconsciously, indirectly used by modelers with a great intuition for modeling or by those who can use the experience, remarks or recommendations of other modelers.

There are many texts in the literature dealing with the topic of the principles of modeling (see the literature cited at the end of this chapter). The importance of these principles varies from one discipline to another (a brief, partial discussion of this is given in, e.g. M. Eronen (2015), T. Bartelborth (2016), I. Peschard (2011), N. Fillion and R. Corless (2014), E. Kummerfeld and D. Danks (2014), J. W. Romeijn and O. Roy (2014), A. Spanos and D. Mayo (2015), and Chapter 3 of this volume). For econometricians, economists, financialists, managers and statisticians, the following list of 60 modeling principles (in short: mp) should be, we hope, both interesting and instructive.

The list contains the following principles:

modeling principles

mp1) a modeler should be accountable for the quality of his model,

mp2) a modeler should respect and use suitable rules of abstraction when searching for appropriate forms of model M, mp3) a modeler should use the modeling rules which he knows best from a theoretical and practical point of view,

mp4) the accuracy of ex post and ex ante prediction based on the model should be high,

mp5) the lists of entities in W_m, $\hat{W}_m$, $\hat{W}_{m,cc}$, $\Phi(W_m)$, $\Phi(\hat{W}_m)$ and $\Phi(\hat{W}_{m,cc})$, as well as W_{nm}, $\hat{W}_{nm}$, $\hat{W}_{nm,cc}$, $\Phi(W_{nm})$, $\Phi(\hat{W}_{nm})$, $\Phi(\hat{W}_{nm,cc})$ should be appropriately adapted to the data available, where Φ denotes features, $\hat{}$ denotes modeled objects, $\hat{W}_m$ the set of modeled material objects and $\hat{W}_{m,cc}$ the set of concomitant material objects included in the model M,

mp6) attention should be paid to ensuring the adaptability of a model to changes in the forms and significance of the objects being studied,

mp7) care should be taken in choosing suitable methods of modeling and technical facilities, in order to achieve the given goals of modeling,

mp8) adherence to modeling rules which are promising or effective,

mp9) use various well-known materials and tools, including the use of atypical data, to check a model's robustness,

mp10) when adjusting a model's form, remember not to lose its interpretative value,

mp11) define a system of indicating problematic regions of $D(M)$, $CD(M)$, to ensure the robustness of M,

mp12) prepare a sufficiently good algorithm describing the modeling process,

mp13) in scientific modeling take into consideration both intuitive, as well as, analytical approaches,

mp14) prefer data-anchored models to overly doxastic theoretical models,

mp15) prefer technology-oriented epistemic aims to ancillary aims,

mp16) anticipate difficulties in modeling connected with the poor quality of observational data, modeling methods, the specific form of M,

mp17) be open to checking antithetical hypotheses, axioms, assumptions, mp18) take care about the suitable form of appearance of objects connected with M,

mp19) use many approaches in the choice of $D(M)$, M, $CD(M)$, $D(T)$, T, $CD(T)$, where T denotes a theory,

mp20) place high importance on model M producing small errors in approximation,

mp21) pay serious attention to arguments refuting M, $M(T)$ or T,

mp22) maximize the number of hypotheses and forms of the model that have been verified,

mp23) respect an appropriate balance of the share of modeled and concomitant objects and their features according to the following binary classifications:

- qualitative, quantitative,
- non-observable, observable,
- random, deterministic,
- non-verifiable, verifiable,

mp24) take into account the basic perceptions, opinions and ideas proposed by other modelers,

mp25) put more faith in the sincere confirmation of a model's validity than in its supposed potential,

mp26) base your modeling practice more on calculations than waiting and seeing what objects "tell",

mp27) do not cheat yourself by using canny methods to attract celebrity while ignoring the main community of modelers,

mp28) do not capitulate in searching for the best model,

mp29) do not present your model as the only means of providing competitive forecasts in a diverse range of scenarios,

mp30) ensure your model's compatibility, comparability and capacity to represent the features of objects well, as well as the ability to generate new models of W_m, W_{nm} and preserve the clarity of semantic interpretations, both of the model's elements and their correspondence to reality, together with a model's internal coherence with the studied W and avoidance of ambiguity,

mp31) take care that your constructed models are: consistent, coherent, complete, determinable, interpretable (both semantically and formally), epistemically valuable, verifiable (with respect to W, $\Phi(W)$, $T(W)$ and $T(M(W))$) and balanced with respect to the qualitative and quantitative features $\Phi(W)$ of the objects W, as well as $M(\Phi(W))$. The model should give accurate approximations of real processes and conclusions based on $T(W)$ which are compact in form, flexible to changes in the form or scale of W, W_{cc}, credible (empirically and theoretically), admissible (logically, mathematically, empirically, computationally, cognitively, theoretically and epistemically), as well as being sufficiently concise and concrete (also comprehensible, complementary or competitive to existing and still unrefuted models or theories),

mp32) take care that the results of your modeling represent the properties of empirical data well (such as symmetry, monotonicity, patterns of data cyclicity, asymmetry or patterns of data synchronicity, diachronicity, changes in and coincidence of signs of the estimates of the parameters of a model and the signs of estimates of the correlations between appropriate explananda and explanans variables corresponding to these parameters, the patterns of the sensitivity of data structures with respect to the results of such operations as scaling, translation, standardization, orthogonalization, orthonormalization, grouping, reduction and interpolation),

mp33) preserve the equality of numerical invariants of empirical data, and the corresponding invariants of modally generated data (predicted or simulated data), where invariance is taken with respect to the results of the seven kinds of operations mentioned above,

mp34) aim to fit, approximate or extrapolate the data as well as possible, or at least recognize and explain the processes or mechanisms governing the phenomena represented by the data gathered,

mp35) aim to give the most probable explanations and accurate predictions of the features of the objects modeled,

mp36) aim to give the most robust explanation and prediction of changes in the features of objects, mp37) aim to give the best description (and explanation) of W, $\Phi(W)$, $T(W)$, $T(\Phi(W))$ from the point of view of how understandable a model is to its users or evaluators,

mp38) aim to make the best possible use of existing theories,

mp39) aim at perfect use of the rules of induction, deduction, reduction and abduction,

mp40) aim at the best use of the principle of limited confidence in the quality of the data and effects of their use in empirical investigations,

mp41) build or formulate decomposable models that make it easier to discover important factors and enable improvement,

mp42) construct models which are empirically reliable and can be used on many occasions,

mp43) be adaptable in cases where the results from modeling are unsatisfactory or the following are inappropriate: methods of observation, ways of measurement, methods for estimating the parameters of the model, means of classifying the objects W, W_m, $W_{m,\ cc}$, W_{nm} and $W_{nm,\ cc}$,

mp44) adopt the research chain: empirical modeling at actual time t_a, empirical theorizing at $t_a + \tau_1$, theoretical modeling at $t_a + \tau_2$, empirical modeling at $t_a + \tau_3$, where $t_a, \tau_1, \tau_2, \tau_3$ denote suitable time limits, differently placed in different research projects,

mp45) prefer well elaborated and verified models to fashionable models,

mp46) be cautious about verifying the epistemic validity of a counterfactually robust invariance relation between the explanandum and explanans factors (due to the epistemic difficulty of differentiating between factors),

mp47) treat the significance and validity of the invariance relation between a given statistical property of the sample set studied and a change in the structure of this set in a careful manner,

mp48) prefer a strategy of fixing computational-algorithmic measures of the robustness of the modeling process rather than random-walk methods of estimating this robustness, mp49) select the model that best fulfills the modeler's chosen aims,

mp50) select the model that best recognizes the true causes of the modeled objects,

mp51) select the model that best represents as many important (also new) properties of the modeled $\hat{W}$, $\hat{W}_m$ and $\hat{W}_{nm}$ as is possible, and as few wrongly recognized properties as is possible, as well as minimizing the number of unrecognized features of the objects studied,

mp52) try to respect all reasonable knowledge about the modeled object and any concomitant objects, especially constraints on information, the applicability of the model to prediction, and simulations conducted on the basis of the chosen model,

mp53) construct a valid model which is constantly verifiable (empirically, formally, experimentally, computationally, descriptively, explicatively, predictively ex post, ex ante, operationally in practice, controllably, manageably, by simulation) and adjustable (under appropriate timing and adaptation rules) according to the assumptions made, which takes into account the influential-confounding factors affecting qualitative or quantitative changes in the objects modeled.

This list of the above 53 principles for modeling reflects many views and opinions, of active modelers (of whom I am one), users and evaluators of models, philosophers of science, as well as some leaders of many disciplines in science and the arts. It is obvious that no modeler can use all of these principles simultaneously. Nobody can be an expert in all of the methods of modeling developed due to natural limits on one's life time and the need to keep up with the ever increasing number of new techniques, facilities and methods. These constraints mean that there are restricted possibilities of assigning, before starting the modeling process, the right methods and principles to the chosen objects of study, and there exists a rather long learning process about the attributes of objects, due to the usually imprecise results of measurement, and observation in general. The choice of suitable principles in a concrete research study will depend on the relevant discipline of science or art, chosen aim of research, chosen kind of object to be studied, experience and modeling skills of the modeler, the quality of data, the planned time-span for modeling, and financial constrains on research.

The last three principles, mp51-mp53, generate many important questions, which should be answered by modelers. These are as follows:

q1) how to include, within a modification of a model, the most important findings resulting from new data, fact-based control of objects, analysis of the final model, conclusions about changes in the structure of W, W_m, W_{nm}, $W_{m,cc}$, $W_{nm,\ cc}$, and attributes $\Phi(W), \ldots, \Phi(W_{nm,cc})$, conclusions about the results of experiments based on the model M, results of method-based control of data quality, changes in $\Phi(M)$, $\Phi(T)$, together with changes in $\Phi(W), \ldots, \Phi(W_{nm,cc})$, and possible changes in predictions, methods of simulation and the results of their use;

q2) How important is the model M, i.e. what is the degree of model M with respect to the following:

- representativeness of the results of modeling achieved according to M, in describing, explaining and predicting the modeled objects $\hat{W}$, as well as the changes that are taking place in $\Phi(M)$ and $\Phi(W)$;
- robustness to changes in the data structure, influential results of observations, as well as the methods of model estimation and prediction used, or the methods of simulation and optimal control;
- how well the model fits the data and the accurateness with which the features of objects are described, explained and predicted,
- sensitivity of performance to changes in the data or their structure, the methods of data analysis, model estimation, prediction, simulation or optimization applied;
- stability of prediction or results of optimization,
- decidability of conclusions about M, T, that is to say M-, T-consistency, completeness, causality, validity and truthness,
- predictability of the phenomena and processes which are to be described and explained by the model, or the relations between $\hat{W}_m$, $\hat{W}_{m,\ cc}$, $\hat{W}_{nm,\ cc}$, where "ˆ" denotes "modeled",
- theoretical and practical validity or relevance of taking into account a structural explanation and prediction of the features $\Phi(\hat{W}_{cc}, \hat{W}_m, \hat{W}_{nm,cc})$, as well as the manageability of achieving the planned research goals,
- presenting a successful solution to a dynamically variable range of research problems,

- valid inclusion of the interaction between internal and external factors which determine the evolution of changes in $\Phi(\hat{W},\ \hat{W}_{cc})$;

q3) What are the necessary and sufficient conditions for successful:

- data mining that enables basic understanding regarding the objects being studied,
- data processing (aggregation, deseasonalization, decyclization, scaling, translation, mean adjustment, demodalization, standardization, integration, cointegration, interpolation),
- modeling of atypical phenomena (physical, chemical, biological, social, economic, psychological, political, historical, cultural),
- prediction ex ante of natural phenomena, or economic and social events and processes,
- model-based decision making and realization of planned aims;

q4) What are good reasons for constructing sensible models and executing successful modeling?

The following are good reasons:

- representing the objects being modeled (e.g. real-factual, historical, artifactual, fictional-virtual),
- describing the objects modeled, as well as objects which are contextually concomitant with them,
- explaining the origins, duration, development and annihilation of the objects modeled, as well as their interaction with other objects and the periodical variability of the attributes of such objects,
- predicting features of the objects modeled (ex post, ex ante), and the forms of the relations between the features of modeled and concomitant objects,
- use of models to induce the discovery of new objects, and to improve the description of features of existing objects,
- broadening and deepening knowledge about the features of objects, modelers and model users learning new skills in modeling,

- assessing the working lifespan of the objects modeled in relation the lifespan of other objects,
- assessing the possibility of using models in forming new classes of models and new objects to be used in practice, as well as in increasing the effectiveness of research,
- testing the quality of prototype products planned for mass use,
- testing the reliability of old or new products used in real life situations,
- testing the reliability of new methods of modeling,
- testing for economic and scientific gains resulting from the use of new models and methods of modeling,
- testing for the existence of research gains resulting from freeing models from being closely related to suitable theories or real world objects,
- testing for the existence of research gains from using models as generators of new models,
- testing the impact of emotions in modeling on the invention of new models and new prototype objects that can be used in science and real life.

Note 4.14. It is impossible to give a general answer to any of the above questions. It is possible to answer a single question related to a chosen modeled object $\hat{W}_i$, chosen model $\hat{M}(\hat{W}_i, \hat{W}_{i,cc})$, and chosen aim of modeling by a concrete modeler. However, this can only be done partially for each question taken individually. A whole variety of research situations will not be covered by such an answer.

The last 14 research situations listed under q4 are related, similarly to the situations listed under questions q1–q3, to our last 3 propositions mp51–mp53 that are not discussed in the literature given at the end of the chapter. We present, moreover, seven further groups of principles that are considered by us to be very important, both practically and epistemologically. These are the principles mp54–mp60 that we list below:

mp54) construct models which maximize the value of a model as measured by its capacity to represent or describe the objects modeled, to interpret $\hat{W}$, $\hat{W}_m$, $\hat{W}_{n,m}$ (classes of modeled objects), to predict future states of $\hat{W}$, $\hat{W}_m$, $\hat{W}_{n,m}$ as characterized by their most important parameters or features, or to facilitate the development of new structures of model (physical or attributive) which could be crucial to successful research and its use,

mp55) form models that: a) maximally fulfill the chosen aims, b) maximally preserve the practical lifespan of a model to represent, interpret, describe, explain, validate and predict,

mp56) form models that: enable epistemic and empiric comparison of their degree using ceteris paribus conditions and the principle of indifference conditions (i.e. evidential symmetry between scientific propositions, sentences that describe $\hat{W}$, $\hat{W}_m$, $\hat{W}_{nm}$ and $\hat{W}_{cc}$),

mp57) form models that respect a chosen (assumed) hierarchy of aims, methods and principles of modeling,

mp58) take into account technical constraints connected with access to sufficiently rich data sets, sufficiently powerful computing facilities and numerically efficient convergent algorithms during selection of $\hat{W}$, $\hat{W}_m$, $\hat{W}_{nm}$ and $\hat{W}_{cc}$,

mp59) do not overestimate the possibility of obtaining optimal or near optimal results from modeling in a reasonable period of time while incurring an acceptable cost for obtaining results in a way that can be easily repeated or reproduced,

mp60) monitor your capacity and will to model, in order not to lose recognition as a good modeler.

4.7. Models and modeling in economics

In this section we aim at presenting some specific problems of modeling in economics. From the 50-ies of the XX c. economists, besides continuing their traditional descriptive-explanatorial approach to the subjects of macro- and microeconomics, have been broadly and intensively developing a formal approach. This uses mathematical language, methods, models and personal computers, which enable the carrying out of complex numerical calculations, simulations and data processing. As in physics, in economics, there is a prevailing tendency to formalize the subjects of research by the use of modern advanced mathematical tools provided by an extremely rich collection of applied and theoretical mathematical models and methods for their estimation. Their use embraces:

- the description, explanation, prediction and optimization of the functioning of complex economic systems (which are much more complex than physical systems),
- a more precise formal interpretation of studied economic events, phenomena and processes, both at micro and macro-scale,
- modeling as a way to solve not only research problems, but also current practical, economic or economic-social problems,
- extensive application of statistical-econometrical methods in predicting the micro and macro properties of the structures of modeled objects $\hat{W}_m$, $\hat{W}_{nm}$,
- advanced specialization in either theoretical or practical research, with a decreasing proportion of researchers doing research in both.

Recently, it has been observed that applied economic research tends to be much better financed than theoretical research. This means that the most talented persons are choosing, during their education process, applications-oriented subjects and, in particular, are learning the skills of empirical modeling and applied informatics. Due to rapid technical progress and the robotization of production lines, as well as advanced economies placing a strong stress on the clean production of energy (particularly in Japan, Germany, France and Great Britain), an applied approach is taking the lead. These changes and the quickly growing economic power of banking systems (especially seen in the growing role of mega investment banks, financial markets and institutions, emergence of new financial products, universalization of commercial banks and the strong influence of large stock exchanges on many governments) have increased the role of the fields of economics dealing with financial phenomena, processes and their links with other domains of economic research.

It is necessary to recall that the concept of "economics" covers all known economic activities, such as: the production, division, exchange and circulation of goods (in space and time, among persons), as well as their accumulation (by physical and legal entities), consumption, investment and gifting. When we use the term "goods", we may think about physical goods (e.g. cars, houses, clothes, computers, furniture), financial goods (e.g. money, commercial papers (bonds, stocks, options, swaps)), artistic goods, scientific goods, medical goods, insurance goods and so on.

From the 70's of the XX c., there has been a clear rise in the role of such capital goods as: human capital (skills, experience, biological and mental energy, accumulated knowledge, passion to learn new skills) and social capital (all kinds of social ties, social activities, social skills, acquirement of social knowledge), as well as authors' rights and intellectual rights to original capital goods. There is a conviction among many (if not most) economists that these concepts are strongly connected, both theoretically and in the process of modeling.

Even a superficial look over famous XIX c. treaties on economics (e.g. books written by A. Smith, D. Ricardo. J. S. Mill, W. Jevons, A. Marshall, J. B. Say, A. Cournot, E. Böhm-Bawerk, J. B. Clark) unveils that they give us economic theories as collections of the author's or their contemporaries' subjective opinions, views, utterances, protocol-observational propositions or sentences about the economic history, phenomena and processes observed by them or their predecessors. In the case of A. Smith, J. B. Say, J. S. Mill, W. Jevons, E. Böhm-Bawerk, the language used is informal and descriptive. Each usually introduced the same theoretical and practical concepts as the other authors using terms with meanings close to the intuitive content known from everyday practice by people living before or during their times.

Due to the fact that in the XIX c. the technical facilities of measurement were very poor by today's standards, the gathering, processing and transmission of data on a massive scale were impossible. Hence, the economists mentioned above mainly used intuitive, theoretical and abstract concepts and sometimes included empirical data as an illustration of their theoretical narration. These data referred to demographics, employment and unemployment, production and trade, as well as government income and expenditure, wages, profits, rent and interest. D. Ricardo used simple algebraic tools and a small system approach to reflect the structural relations between the main aggregates of an economy. A. Marshall, in turn, uses mathematical analysis and elementary geometry to explain his microeconomic conjectures and mostly theoretically oriented theses. The most advanced use of mathematical tools was applied by A. Cournot, a French mathematician, who was the first to present a mathematical theory of exchange, value, monopolies, competition and taxes. He used, in a very clear way, both differential and integral calculus. J. B. Clark's vivid debates with E. Böhm-Bawerk on some points of the theory of capital were not only descriptive, but also analytical and used mathematical ideas. W. Jevons, who was simultaneously an economist and logician, preferred a very clear logical presentation of his views on political economy and the theory of markets.

For the purposes of this book, the relationship between the concept of a theory and its model will be discussed on the basis of the unusually influential book "An inquiry into the nature and causes of the wealth of nations" written by A. Smith. The author used empirical terms which were used by the general populace, including the proper names of famous kings known from historical records, army commanders, great reformers of legal and socio-economic systems, religious leaders, chiefs of financial institutions, as well as geographical names, including the names of nations and countries, as well as the names of professions, companies, corporations and craft workers' organizations. Smith's theses are predominantly empirical and expressed in mostly empirically sounding microeconomic and macroeconomic terms. In his book, we can also find such theoretical terms as demand and supply for goods and labour, as well as the notion of capital, wealth of a nation, interest, idleness, cause of wealth, competition, philosophy, labour, division of labour, the governing principle of such division, risk, wants of mankind, wealth, ontology, faith, tendency, natural progress, technical progress and natural rate.

When reading this book, one can easily see Smith's erudition regarding the history of mankind till his times, his thoughtful penetration of the possible causes of the wealth of nations, the causes of the most important political, military, economic, cultural and religious events in both the UK and Europe. A. Smith was also a master of unambiguous explanations of common or specific opinions and the views of other historians and economists from the UK and France, both his predecessors and his contemporaries. His summary of these opinions is compact, very clearly written and very rich factually.

We wish to stress that Smith's economic theory of the wealth of nations is empirical, historically and cognitively well written, descriptive, qualitatively explicative, and informal. The language used to present his theory, L_T, is a mixture of the following kinds of languages: the English language, L_E, the language of history, L_H, the language of economics, L_{Ec}, the language of descriptive statistics, L_{St} and the language of arithmetic, L_{Ar}. Denoting these languages by $\hat{L}_E$, $\hat{L}_H$, $\hat{L}_{Ec}$, $\hat{L}_{St}$, respectively, their union takes the form $\hat{L}^{AS} = \hat{L}_E \cup \hat{L}_H \cup \hat{L}_{Ec} \cup \hat{L}_{St}$, where the set $\hat{L}^{AS} \subset L$, $L = L_E \cup L_H \cup L_{Ec} \cup L_{St}$, and AS denotes A. Smith. However, formal definitions of his theoretical concepts require more groups of entities (for a specification of these groups, see e.g. the chapter "On theories and theorizing" included in this book). From the point of view of a formal definition of his theory,

A. Smith did not explicitly mention, for instance, any logical axioms, clearly defined rules for reasoning, formal definitions of the concepts used, theorems or formally stated criteria for evaluation. The same can be said about similar requirements resulting from the general schematic definition of economic theory included in this text (see Definition 7).

Does this mean that Smith's theory T^{AS} is non-refutable (non-confirmable) empirically? In my opinion it is, at least, partially confirmable (e.g. his theory of economic growth based on data from Poland between 1993 and 1998, a period of economic transition). Empirical testing based on Polish quarterly data was carried out using a model describing Polish economic growth in this period. The determinants of this growth were taken from the theory T^{AS}. Growth was expressed in the form $Y^{\circ}(t) = \triangle Y(t)/Y(t-1)$, where $\triangle$ denotes the first difference operator defined as $\triangle Y(t) = Y(t) - Y(t-1)$, t denotes the time period, $Y \equiv GDP$, where GDP stands for Gross Domestic Product (monetary value). A detailed description of the data set, the statistical definition of Y and the explanatory variables of economic growth is included in W. Milo [(2002), Chapter 4]. The list $\mathbb{X}$ of determinants given in the form of a row vector is as follows:

$$\mathbb{X}(t) \stackrel{\text{def}}{=} (DECOMP(t), TYSPRH(t), SIS(t), DEFERW(t), RFPR(t), WFQQ(t), CPRR(t), SPEM(t)), DEMPN(t)) \stackrel{\text{def}}{=} (X_1(t), X_2(t), X_3(t), X_4(t), X_5(t), X_6(t), X_7(t), X_8(t), X_9(t)),$$

where, appropriately, $X_1(t) \equiv DECOMP(t), \ldots, X_9(t) \equiv DEMPN(t)$.

The particular variables (factors) names denote:

$DECOMP$ – degree of competition in a given economy,

$TYSPRH$ – type of sectoral hierarchy of productivity,

SIS – structure of social institutions,

$DEFERW$ – degree of fervor in performing work,

$RFPR$ – rewards according to factor productivity,

$WFQQ$ – work force quality and quantity,

$CPRR$ – rate of capital profit,

$SPEM$ – spatial expansion of the market,

$DEMPN$ – density of the network of trade points.

Statistical computations were carried out based on a strategy of building models starting from one factor and successively increasing the number of factors to two (all pairs), to three (all triads), and so on, using the t-test, F-test, Chow-Fisher stability test, Kendall-Hellwig coincidence of signs test, stationarity test and co-integration test.

The prototype linear model of the relationship between the rate of economic growth $Y^{\circ}(t)$ for the Polish economy and the determinants described by Smith is as follows:

$$\overset{\circ}{Y}(t) \overset{\text{def}}{=} \sum_{i=1}^{9} \beta_i X_i(t) + E(t) = \beta \mathbb{X}(t) + E(t), \beta \in \mathbb{R}^9,$$

where E (t) denotes the specification error for the linear model based on the theory T^{AS}, and $\beta = (\beta_1, \ldots, \beta_9)$ is the row vector of parameters.

In practical applications, we also used the following, more convenient, form of the right-hand side

$$X_1^{\beta_1} \ldots X_9^{\beta_9} E(t),$$

which, after logarithmic transformation, takes the form

$$\sum_{i=1}^{9} \beta_i x_i(t) + e(t),$$

where $x_i(t) \overset{\text{def}}{=} \ln(X_i(t))$, $e(t) = \ln E(t)$.

Based on this expression, we defined changes in the value of the growth rate either as

$$\overset{\circ}{Y}(t) \overset{\text{def}}{=} \sum_{i=1}^{9} \beta_i x_i(t) + e(t),$$

or

$$\overset{\circ}{Y}(t) = \beta_0 + \sum_{i=1}^{9} \beta_i x_i(t) + e(t), \beta_0 \in R.$$

Note 4.15. Estimation of β_0 and β_i, $i = 1, 9$, on the basis of the sample data $\{y^{\circ}(t), y_i(t), i = \overline{1, 9}, t \in [1993.3 - 1998.3]\}$, was done using the Gauss-Legendre method. The data were taken from the archives of GUS and NBP (the State Statistical Office and the National Bank of Poland). This period of the development of the Polish economy was subject to turbulent changes connected with the first phase of transition from a centrally planned socialist economy to a market economy. Deregulation, the so called “wild” privatization where many parts of the Polish economy were sold to foreign owners, brought the necessary foreign financial capital to guarantee the exchangeability of the Polish currency, the zloty, to dollars, the British pound and other currencies. This was done at the cost of the bankruptcy of a large number of firms (which were further often sold for sometimes 15% of their true value). This caused a very high unemployment rate and a great wave of economic emigration in search of work abroad.

For obvious reasons, Smith's theory does not consider factors specifically arising in a period of transition from a socialist economy to a market economy highly influenced by international corporations, companies and investment banks. Neither does it contain any elements that reflect the impact of delayed investments in at least physical and financial capital, which could be observed in the production goods markets and labour markets, as well as in economic growth.

Using the approach described above, we treat our estimates of the growth rate based on this model as an empirical realization of Smith's theory of economic growth. They were generated based on the following description of the model based on T^{AS}

$$MAST \stackrel{\text{def}}{=} (\hat{L}^{AS},\ \hat{L}^{MAST}, \hat{F}\left(Y, \mathbb{X}, \beta, E\right), Cn(\hat{F}), DEF(Y, \mathbb{X}, \beta, E),$$

$$Cn(DEF(Y, \mathbb{X}, \beta, E), RR, EC^{e}),$$

where:

$\hat{L}^{AS}$ denotes the language of Smith's theory,

$\hat{L}^{MAST}$ is the language used in the presentation and discussion of the model $MAST$,

$\hat{F}\left(Y, \mathbb{X}, \beta, E\right) \stackrel{\text{def}}{=} (Y^{\circ}\left(t\right) = \beta_0 + \sum_{i=1}^{9} \beta_i x_i\left(t\right) + E\left(t\right)$, $\beta \in \mathbb{R}^{10}$, $x_i\left(t\right) \in R$, $E \in N(0, \sigma_E^2)$, where R is the set of reals, $\mathbb{R}^{10}$ the space of reals with $\dim\left(\mathbb{R}^{10}\right) = 10$, E has a normal distribution with expected value zero and variance σ_E^2,

$DEF(Y, \mathbb{X}, \beta, E)$ denotes the list of definitions of the symbols $Y, \mathbb{X}, \beta, E$, both formal and semantic,

$Cn(\hat{F})$, $Cn(DEF(Y, \mathbb{X}, \beta, E)$ are the lists of consequences of assuming particular definitions for the symbols $\hat{F}, Y, \ldots, E$,

RR denotes the rules of reasoning applied (induction, deduction, laws of probability and statistics, econometric and economic rules),

EC^{e} denotes the evaluation criteria used to assess the econometric model $MAST$, i. e. the following criteria: stationarity, significance, stability testing, turning points index, coincidence index, multiple correlation coefficient, ill-conditioning index.

We used the following ex post predictors:

$\widehat{Y^{\circ}}(t) \overset{\text{def}}{=} < \widehat{\beta}, \mathrm{x}(t) >$, $\widehat{Y^{\circ}}(t_{\tau}) \overset{\text{def}}{=} < \widehat{\beta}, \mathrm{x}(t-\tau) >$, where $< .,. >$ denotes the scalar product of vectors with lags $\tau = 1, 2, \ldots, 6$, $\mathrm{x}(t-\tau) \equiv (x_1(t-\tau_1), \ldots, x_9(t-\tau_9)$, $\tau_i \leq 6$.

Implicitly, the description of the model $MAST$ may include axioms and theorems from probability, statistics, econometrics and mathematical economics. For empirical computations and interpretations, we must respect and use all the information about the quality of the data available, as well as the precision of estimation and prediction. In our econometric computations, we were not able to find reasonable statistical measures for the four factors described by Smith and called by us $TYSPRH, SIS, DEFERW$ and $SPEM$. From the rest of the list of $\mathbb{X}$ variables, it turned out that only the variables $WFQQ, PFRR$ and $CPRR$ taken with suitable lags were found to be statistically significant factors for explaining Polish economic growth in the period 1993.3–1998.3. This means, therefore, that Smith's theory of economic growth is only partially confirmed statistically, because we did not include in our model four factors of economic growth described by Smith. However, even given this incomplete version of the model, the proportion the variation of empirical growth in the Polish economy explained by the model was very high (97%), and this version of the model has clear cognitive, descriptive, explanatory and predictive value.

It is an open question as to whether the discussed (or perhaps another) version will conserve the mentioned roles and features based on subsets of these 9 factors of growth when one considers the periods 1999–2004, 2005–2007, 2008–2016, which are highlighted here in order to reflect the impact of Poland joining the EU, as well as the world financial crisis.

For readers interested in the empirical performance of theories of economic growth reconstructed from the books of D. Ricardo, J. S. Mill and A. Marshall based on empirical data for the Polish economy in the period 1999.3–1998.4, we recommend reading chapters 3, 4 of the monography by W. Milo, et al. (2002) previously cited.

In this place, it is important to underline that the lists of factors used by Ricardo, Mill and Marshall have only one common factor of Poland economic growth, the rate of profit at the macro-scale. In the case of Ricardo's theory, the most influential factor is productivity growth in logarithmic form, and the second most influential factor is technical progress in the Polish economy. The rate of growth of circulating capital was less influential. In general, the historical level of the proportion of variation of growth explained, 99.5%, was also stable over time. J. S. Mill's ten determinants of growth explained a stable proportion of 83% of the variation in growth in the Polish economy with productivity, together with wage and trade conditions, being the most influential factors. L. Walras' theory fitted the data poorly even using the best version of such a model. We also considered models based on A. Marshall's theory of economic growth. The best dynamic version explained 66% of the variance in economic growth, and the most significant factors were: delayed interest rate, technical progress, profits (net), taxes and capital of firms.

The following are general remarks on these empirical models of growth built according to the theories of Smith, Ricardo, Mill, Walras and Marshall:

- these empirical models fit the historical data, 1993.3 – 1998.4, on the growth of the Polish economy well, except for the model based on L. Walras' microeconomic theory,
- the considered theories were partially confirmed statistically, at least based on data regarding economic growth in Poland,
- improvements would be possible if:
  - the statistical data are improved in quality, and other factors of growth are included,
  - we could define methods for measuring those factors that until now have not been measured statistically, but seem to be very important,
  - longer time series (for the Polish economy) in order to study hysteresis effects,
  - missing data points could be well estimated,
  - nonlinear relations are also investigated,
  - feedback relations between explanans factors and explanandum are estimated and tested,
  - system methods of estimation and prediction are used.

4.8. Final remarks

This chapter contains less formal continuation of the discussion carried out in Sections 3.3, and 3.4 of Chapter 3. It is hoped that the included long list of many classifications of kinds of models, more specific forms of models, modeling principles, will be a useful foundation for making comparison between different models useed by economists. In Section 4.7 we have presented the example how to apply some of the concepts discussed in Chapters 3, 4 and 10.

Bibliography

Adomian, G. (1983). *Stochastic systems.* Academic Press, New York.

Aduriz-Bravo, A. (2015). Recent semantic developments on models. *Science Education*, No. 24, pp. 1245–1250.

Anderson, C. (2013). Fallibilism and the flexibility of epistemic models. *Phil. Studies*, Vol. 167, No. 3, pp. 597–606.

Balzer, W., Moulines, C. U. (1980). On theoreticity. *Synthese*, Vol. 44, pp. 467,-494.

Bartelborth, T. (2016). How strong is the confirmation of a hypothesis by significant data? *J. Gen. Phil. Sc.*, Vol. 47, No. 2, pp. 277–291.

Benioff, P. (2002). Towards a coherent theory of physics and mathematics. *Found. Phys.*, Vol. 32, No. 7, pp. 989–1028.

Berger, J. O., Berry, D. A. (1988). Statistical analysis and the illusion of objectivity. *American Scientist*, No. 76, pp 159–165.

Bianchi de, S. (2016). Which explanatory role for mathematics in scientific models? Reply to "The Explanatory Dispensability of Idealizations". *Synthese*, Vol. 193, No. 2, pp. 387–401.

Blaug, M. (1992). *The methodology of economics: or how economists explain.* CUP, Cambridge.

Boumans, M. (2002). *Calibration of models in experiments.* "Model-based reasoning", eds. L. Magnani, N. J. Nersessian Kluwer N.Y., pp. 75–93.

Brössel, P., Huber, F. (2014). Bayesian confirmation: a means with no end. *The British Journal for the Philosophy of Science*, Vol. 66, No. 4, pp. 737–749.

Bunzl, M. (1984). A causal model for causal priority. *Erkenntnis*, Vol. 21, pp. 31–44.

Cambell, S., Franklin, J. (2004). Randomness and the justification of induction. *Synthese*, Vol. 38, pp. 79–99.

Cartwright, N. (1979). Causal laws and effective strategies. *Nous*, Vol. 13, pp. 419–437.

Constantini, D., Garibaldi, U. (2004). The Ehrenfest fleas: from model to theory. *Synthese*, No. 139, pp. 107–142.

Czerwiński, Z., et al. (1987). *Ekonometria.* PWN, Warsaw.

Eddington, D. (1992). Validity, uncertainty and vagueness. *Analysis*, No. 52, pp. 193–204.

Ellis, B. (1966). *Basic concepts of measurement.* CUP, Cambridge.

Eronen, M. (2015). Robustness and reality. *Synthese*, Vol. 192, No. 12, pp. 3961–3977.

Fetzer, J. H., Nute, D. E. (1980). A probabilistic causal calculus: conflicting situations. *Synthese*, Vol. 44, pp. 241–246.

Fillion, N., Corless, R. M. (2014). On the epistemological analysis of modeling and computational error in the mathematical sciences. *Synthese*, Vol. 191, No. 7, pp. 1451–1467.

Frejd, P., Bergsten, C. (2016). Mathematical modelling as a professional task. *Educ. Stud. Math.*, No. 91, pp. 11–35.

Freundlich, Y. (1974). Objects and their attributes: a physicist's point of view. *Foundations of Physics*, Vol. 4, No. 1, pp. 1–8.

Gilbert, C. L. (1989). *Economic theory and econometric models.* Paper No. 194, Queen Mary College ED. Univ. of London.

Glennan, S. S. (1996) Mechanisms and the nature of causation. *Erkenntnis*, Vol. 44, pp. 49–71.

Grundman, T., Horvath, J., (2014). Thought experiments and the problem of deviant realizations. *Phil. Stud.*, No. 70, pp. 525–533.

Handler, E. W. (1982). The evolution of economic theories. A formal approach. *Erkenntnis*, Vol. 18, pp. 65–96.

Hansen, L. P., Sargent T. J. (2008). *Robustness.* PUP, Princeton.

Hendry, D. F. (1995). *Dynamic econometrics.* OUP Oxford.

Hoover, K. D. (2001). *Causality in macroeconomics.* CUP N.Y.

Hoover, K. D. (2015). The ontological status of shocks and trends in macroeconomics. *Synthese*, 192(11), 3509–3532.

Huang, C. Y. (2012). To model or not to model. *Int. J. Forec.*, T28, pp. 497–506.

Hudson, R. R. (2014). *Seeing things.* OUP Oxford.

Illan, P., Williamson, J. (2012). What is a mechanism? *Europ. J. Phil. Sc.*, T2, pp. 119–135.

Ionnides, J. P. (2005). Why most published research findings are false? *PLOS Medicine*, No. 2, pp. 696–701.

Irvine, E. (2014). Models, robustness and non-causal explanation. *Synthese*, Vol. 192, No. 11, pp. 3509–3532.

Jacobs, J. P. et al. (2012). Information, data dimension and factor structure. *J. Multiv. Anal.*, No. 106, pp. 80–91.

Klein, L., et al. (1999). *Principles of macroeconometric modeling.* North Holland Amsterdam.

Jousselme, A., Maupin, P. (2012). Distance in evidence theory. *Int. J. Appr. Reas.*, No. 53, pp. 118–145.

Kruse, M. (2000). Invariance, symmetry and rationality. *Synthese*, Vol. 122, pp. 333–357.

Kummerfeld, E., Danks, D. (2014). Model change and reliability in scientific inference. *Synthese*, Vol. 191, pp. 2673–2693.

Kuorikoski, J., et al. (2010). Economic modelling as robustness analysis. *Brit. J. for the Phil. Sc.*, No. 61, pp. 541–567.

Lewins, R. (1966). The strategy of model building in population biology. *Amer. Scientist*, No 54, pp. 421–431.

Lewis, C. T., Belanger, C. (2015). The generality of scientific models: a measure theoretic approach. *Synthese*, Vol. 192, pp. 269–285.

Masterton, G. (2014). What to do with a forecast? *Synthese*, Vol. 191, pp. 1881–1907.

Matthewson, J., Weisberg, M. (2009). The structure of tradeoffs in model building. *Synthese*, Vol. 170, pp. 169–190.

Milo, W. (1995). *Stabilność i wrażliwość metod ekonometrycznych.* WUŁ, Łódź.

Milo, W. (1990). *Odporność w ekonometrii.* WUŁ, Łódź.

Milo, W. (2013). Losowość a chaotyczność. *Przegl. Statyst.*, RLX, No. 4, pp. 425–445.

Morrison, M. (2013). Values and uncertainty in simulation models. *Erkenntnis*, Vol. 79, pp. 939–959.

Peschard, I. (2011). Making sense of modeling: beyond representation. *Europ. J. Phil. Sc.*, No. 1, pp. 335–352.

Quine, W.O. (1981). *Theories and things.* HUP, Cambridge M. A.

Raerinne, J. (2013). Robustness and sensitivity of biological models. *Phil. St.*, No. 166, pp. 285–303.

Romeijn, J., Roy, O. (2014). Radical uncertainty: beyond probabilistic models od beliefs. *Erkenntnis*, Vol. 79, pp. 1221–1223.

Schaffer, J. (2016). Grounding in the image of causation. *Phil. St.*, No. 173, pp. 49–100.

Schmidt, H. (1978). Can an effect precede its cause? *Found Phys.*, Vol. 8, No. 5/6, pp. 463–480.

Scholl, R. (2015). Inference to the best explanation in the catch – 22. *Europ. J. Phil. Sc.*, No. 5, pp. 89–110.

Smith, M. (2014). Evidential incomparability and the principle of indifference. *Erkenntnis*, Vol. 80, No. 3, pp. 605–616.

Sokolowski, J., Banks, C. (2009). *Principles of modeling and simulation: a multidisciplinary approach.* Wiley, Hoboken N. J.

Spanos, A., Mayo, D. (2015). Error statistical modeling and inference. Where methodology meets ontology. *Synthese*, Vol. 192, No. 11, pp. 3533–3555.

Spohn, W. (2014). The epistemic account of ceteris paribus conditions. *Europ. J. Phil. Sc.*, Vol. 4, pp. 385–408.

Sprenger, J. (2016). The probabilistic no miracle argument. *Europ. J. Phil. Sc.*, Vol. 6, pp. 173–189.

Stegenga, J. (2009). Robustness, discordance and relevance. *Phil. Sc.*, Vol. 76, pp. 650–661.

Symons, J., Boschetti, F. (2013). How computational models predict the behavior of complex systems. *Found. Sc.*, Vol. 18, pp. 804–821.

Tohme, F., Crespo, R. (2013). Abduction in economics. *Synthese*, No. 190, pp. 4215–4237.

Veaux, R., Hand, D. (2005). How to lie with bad data. *Statist. Sc.*, Vol. 20, No. 3, pp. 231–238.

Wasserman, R. (2016). Theories of persistence. *Phil. Stud.*, No. 173, pp. 243–250.

Welfe, W. (2013). *Macroeconometric models.* Springer, Berlin.

Werndl, Ch. (2013). On choosing between deterministic and indeterministic models. *Synthese*, Vol. 190, pp. 2243–2265.

Wimsatt, W. (2007). *Re-engineering philosophy for limited beings.* HUP, Cambridge M.A.

Woodward, J. (2006). Some varieties of robustness. *J. Econ. Method.*, No. 13, pp. 219–240.

Woodward, J. (2014). Methodology, ontology, interventionism. *Synthese*, Vol. 192, pp. 3577–3599.

Worrall, J. (2007). Why there is no cause to randomize. *Brit. J. Phil. Sc.*, No. 58, pp. 451–488.

Chapter 5

On expectations. Part I

Summary

This chapter and the next two of them concern very important economic category of expectations without which it is impossible to make economic research on the micro, meso and macro level. It is now commonly used in economic modeling, theorizing, forecasting.

The category of expectations is interdisciplinary and was already known in antiquity. In this chapter, our attention is focused on two kinds of expectations, rational and rational-adaptive expectations. Our aim is to review the meanings of different forms of expectations, as well as to analyze some consequences of choosing these forms in scientific studies. However, the problems of modeling and forecasting expectations have been excluded to keep the chapter concise.

Keywords: expectations, rational expectations, adaptive expectations, rationality, adaptation, rationalism.

5.1. Introduction

In English the word "expectation" means[1] psychological states of belief, hope, looking forward to something or not wondering whether something will happen in the future because it is likely to happen, ought to happen due to our planned actions or probable for some other reason. This something may be an event or events, a goal or goals, a task or tasks, a phenomenon or

[1] Longman Dictionary of Contemporary English (1955). Longman G. L. Harlow, England.

phenomena, a situation or situations, a process or processes. Thus, this something consists of objects of one's expectations. The list of such objects may be lengthy and depends on one's needs or imagination.

Any analysis of expectations needs to answer the following questions:

q1) what are the objects of expectations?

q2) who is expecting?

q3) who is expecting what?

q4) who is expecting what and when?

q5) who is expecting whose expectations of what?

q6) who is expecting whose expectations of what for what purpose(s)?

A person having or expressing expectations will be called an expector, er_j, $j \in J$, and an object es_k, $k \in K$ of expectations – an expectatis, where J and K denote subsets of the set of natural numbers N. Both expectors and expectatises exist at time $t \in T$, where T is a set of numbers, where $T \subseteq N,\ R^+,\ R,\ C$ as appropriate and $R,\ C$ are the sets of real and complex numbers, respectively. It is assumed that:

- the sets of expectors and expectatises are not invariant in time, space or any structure, in particular economic structures,
- there are interactions and interconnections between the elements of the set $\{er_j\}$ itself, the set $\{es_k\}$ itself and/or between elements from both sets,
- each element, subgroup of both $\{er_j\}$ and $\{es_k\}$ or the sets as a whole have a well defined, and possibly observed and measured, feature, or finite sets of features,
- there are regularities (at least statistical) in the form of the expectations $E_{jk}(t)$, $E_{j.}(t)$, $E_{.k}(t)$, $E_{..}(t)$, $E_{ijk}(t)$, prescribed to $i,\ j,\ k$, where dot **.** denotes expectations corresponding to the whole set of objects and/or persons as appropriate, and $i \in I$ denotes professional expectors;
- the lists of these expectations or prediction rules may be extended to cover regularities which are discovered or obtained through the acquiral of information from other expectors by professional expectors or inquiry, either open or secret.

The set of expectors $\{er_j\}$ includes: e.g. ordinary individuals, households, members of households, consumers, farmers, workers, managers, investors, owners, tenants, market makers, buyers, sellers, decision makers, firms, banks, insurers, politicians, criminals, physicians, journalists, artists, soldiers, policemen, sportsmen, educators, scientists, religious people and their leaders.

The set of expectatises $\{es_k\}$ includes, among other things, a happy life, attractive profession, success in private and professional life, sufficient household income, a high level of self-evaluation with respect to other consumers/neighbors, satisfaction in investment activities, growth in investment, riskiness of exceeding a safe amount of debt, fears of the deconcentration of capital, fears of growing rent, fears of nationalization, fears of growing price inflation, a decline or increase in interest rates, overliquidity of the banking sector, underliquidity of a bank, currency crisis, financial crisis, banking sector crisis, economic crisis, cultural crisis, trade cycle, political cycle, labor market disequilibrium, rise in unemployment rate, outbreak of war, outbreaks of social riots, growth of illegal activities, growth of epidemics, increasing lifespan, changes in population structures, aging of the population, growing secularization, problems resulting from increasing globalization, influence of competition, decline of innovation.

It is now easily seen how to pose other important questions, such as:

q7) whose expectations are the most influential for whom?

q8) which expectations are the most influential for whom?

q9) what are the ways of reinforcing given expectations?

q10) what are the ways of weakening given expectations?

q11) what are the language-semiotic forms of expectations?

q12) what are the philosophical ways of approaching expectations?

q13) what are the ways to model expectations?

In this chapter, attention is focused on the description and discussion of qualitative problems connected with rational expectations and one of their particular forms, namely adaptive expectations. This presentation is based on concepts discussed by leading philosophers, natural scientists and, in modern times, by engineers. For interested readers, a recommended list of books dealing with the issues considered here is given at the end of this chapter.

Section 2 is devoted to concepts of rationality and Section 3 deals with rational expectations. Our presentation of these concepts is based mainly on the literature cited at the end of this chapter.

5.2. Concepts of rationality

The adjective "rational" originates from the noun "ratio" in Latin (in Greek: "$\nu o\tilde{\nu}\varsigma$") which means: reason, basis, consideration. Various philosophical meanings were established in different philosophical approaches to knowledge, reasoning and investigations. From a historic-philosophical point of view, the first form of rationalism was metaphysical rationalism. Its archetypical form was Heraclitus' approach to $\lambda o\gamma o\varsigma$ as the pra-origin of everything in the universe, including human knowledge.

The same may be said about Pythagoras' teaching concerning harmony in the universe, and Anaxagoras' concept of Reason ($\nu o\tilde{\nu}\varsigma$), which is an omnipresent establisher and maintainer of the cosmic order. They believed in and therefore expected and propagated rationality, because they recognized (acknowledged) rationality by assuming the existence of "$\nu o\tilde{\nu}\varsigma$, $\lambda o\gamma o\varsigma$" as being the metaphysical structure of our physical reality which is fully recognizable. Plato and Aristotle presented a more developed version of this form of rationality. Plato asserted that the world (universe) is rational, because it is a reflection of an eternal, invariant sphere of ideas, which are transcendent with respect to the world. So the sphere of the material world is rational, because the sphere of ideas is rational and intelligible. In his dialogue "Timaeus", he says that the material world "has been created according to a model[2] which is subject to Reason and Mind, and so is in itself Reason and Mind".

Aristotle did not share Plato's view that the ontic ground of the world's rationality and cognitivity are transcendent ideas, but he considered the ground (of being) to be the form $\mu o\rho\phi\eta$ $\sigma\chi\tilde{\eta}\mu\alpha$ which is present in each individuum, each thing, each being and its essence to be the primary substance of this form. It is important to say that according to Aristotle, this form is the thing that originates (arises) in something else, either according to non-human nature, to the creative actions of human beings, or to a faculty of being able to act. From the point of view of today's world, we could say that, e.g., robots and automats have such a faculty. The form enables us to recognize a given thing, and to describe its necessary and sufficient causes. According to Aristotle, rationality has its source in the forms

[2] The same belief is now shared by modern model builders in, e.g. physics, chemistry, engineering, and even economics.

observed in the cosmos and the principles of organizing forms, to which the principles of causality, telosity ($\tau\epsilon\lambda o\varsigma$), i.e. aimfulness, belong. His theory of rationality is metaphysical, methodological, cognitive, conceptual and linguistic-semiotical. He linked scientific investigations with syllogistic type procedures of proof and a tendency to synthesize and generalize existing knowledge. Aristotle's rationality is based on his primary philosophical principles of identity, non-contradiction, and sufficient reason that constitute the mental rules of reasoning based on rationality. From his times until the present day, mindful people have asked themselves whether rationality excludes off-rational learning, cognizance, and, for a certain period, improbable theories. Accidental inventions and discoveries show that the answer to this question may by positive. As M. Grabman (1909–1911) noticed, the long-standing early scholastic principle of "sola ratione" played an important role in many, not only philosophical, investigations in the Middle Ages and Renaissance.

The influence of F. Bacon and G. Galileo on the progress of science was important in many respects. The origins of modern concepts of rationality lie in their linking metaphysical rationality to rationality based on empiricism. A greater influence was exerted by R. Descartes. Knowledge is, according to him, based on the metaphysical principles of knowledge accumulation that fulfill conditions requiring absolute clarity and deductive chains leading to the clearest conclusions. "Cogito ergo sum" is the "fundamentum inconcussum". Conceptual rationality needs "clara et distincta perceptio", which is clearly seen in Descartes' writings. He also developed cognitive rationality (intuitive cognition as the basis for episteme – knowledge or understanding), methodological rationality (strict use of logical and mathematical rules). His methodological rationality corresponds to a large degree with Pythagoras' concept of the mathematical structure of the material world. Moreover, his method of "mathesis generalis" uses the category of order (ordo). This was taken from analytical geometry, which was developed by him, as well as a priori and innate concepts related to knowledge based on both autonomous and empirical reasoning. This category of order covers both cause-effect relations and the principles of natural laws. B. Pascal went further in his interpretation of rationality by saying that rationality is not a feature of reality, but rather a feature of knowledge and science. In turn, B. Spinosa, considered the world to be fully rational and to better convince readers he used Descartes' type of "more geometrico".

A particularly broad understanding of rationality is seen in G. W. Leibniz's unusually inventive ideas and theories. His semiotic studies (Meditationes de cognitione, veritate et ideis) gave him the possibility of objectifying criterion for clarity and improving procedures of proof. Leibniz's project of a "lingua universalis" had the goal of freeing reasoning from making errors and improving tools for acquiring new knowledge and communication. It should be stressed that his metaphysics is monadological, and ontic structures, together with the principles of identity, non-contradiction, and sufficient reasoning are the grounds for making progress in science. Searching for the innate principles of mental ideas and subjects of intellectual ideas is, according to Leibnitz, the proper way of carrying out philosophical investigations. Reflection is a primary cognitive power. No fact or saying is really valid if it does not fulfill the principle of sufficient reasoning. The causes of results are often unknown. The criterion of sufficient reasoning is required both as a principle based on our surrounding reality and as a principle based on cognition and action. Thus, Leibniz was a panlogician, i.e. for him the whole of reality is rational and discoverable, so any studies that respect his methodological, metaphysic and semiotic principles are rational and thus the results and objects of studies are also rational. Leibniz's pupil[3] C. Wolff (1729) was probably the first to give us, in Descartes' style, an independent presentation of ontology and metaphysics. According to Wolff, only that which is defined and determined by its essence and fulfills the principle of sufficient reasoning is rational.

I. Kant (1781, 1787) proposed a new concept of rationality as an approach to understanding reality and applying mathematical methods to a philosophy of knowledge and metaphysics. His concept of rationality is strictly epistemological, i.e. negates the identity of a logician's nature of mind, as well as the subject of analysis as an object of ontology. Kant's rationalism is based on the concept of an active subiectum, i.e. a person who is consciously acting and imposing his intellect's Verstandesbegriffe and principles. A scheme of concepts is imposed on outside impressions that helps the subiectum to synthesize them as an object that is observed/experienced. Kant synthesized the empirical approach of D. Hume (Leibniz gave a yet greater weight to a priori rationality). He distinguished, on one side, analytical and synthetic propositions, and on the other, a priori and a posteriori propositions. A priori propositions do not need to be observed/experienced. A posteriori propositions require experience, observation and/or induction. Kant considered that the beginning of cognizance and knowledge lies in the sensual experience of an investigator. The temporal and spatial order is universal and a priori given to a human observer, i.e. bearer of sensual impressions and intellectual intuition. He distinguished 24 conceptual categories, among which the most important was the category of causality. Knowledge covers

[3] See B. Paź (2007, T. 8 p. 604).

phenomena, and not cognizable noumens, i.e. things alone in themselves. When making distinctions between quaestiones facti et juris, he states that the theory of knowledge deals more with answering the second group of questions. It is important to recall that he differentiated deduction from Ableitung (mapping). He understood the second as logical deduction of one sentence from another and the first as deduction as it was then used in legal practice.

Note 5.1. Both a priori and a posteriori propositions are used in economic language, theories and models. They are commonly abstract and hypothetical. Since expectations are prescribed to men's feelings and thoughts, they have a variable character in the dimensions of time and particular human beings. It should be stressed that during Kant's lifetime, experimental psychology and economics did not exist.

Fichte rejects[4] Kant's concept of things being simply themselves. The conscience is dependent on other things and knowledge is based on transcendence. He shares the view that deduction is the only source of necessary truths of knowledge. The principle of the prime importance of the ego is treated as an axiom in his writings.

According to Hegel, being and mind are always changing in regular ways that tend in some direction. This means that the transcendental idealism of Hegel often is called logical, although it is dialectally logical. According to Hegel, all reality is rational, necessary and dialectically logical. This means that his approach to cognizance is radically rationalistic and idealistic. As he understood, the implication relation is not traditionally logical, but is a special intentional relation based on the dialectic principles of the union of contractions and development by successive negation. Hegel's reality is historical with a principle of a built-in period of embracement and this view has proven to be influential in the humanities, especially in Marxist approaches. However, he admits the possibility of objectivity without a historical perspective, but objectivity needs the transcendental perspective of an Absolute Spirit, i.e. the last phase of cosmic evolution. Some roots of Hegel's tradition are seen in Dilthey's hermeneutics. The roots of Kant's ideas, in opposition to Hegalian thought, were revived and developed in neo-Kantian schools of thought (e.g. H. Cohen, P. Natorp, W. Windelband, H. Rickert and L. Nelson).

Belief in the full rationality of reality (i.e. metaphysical rationality) was common to Hegel and the XVII c. rationalists. Hegel understands the rationality of the world as the existence of Reason (Vernunft), which controls

[4] See J. Woleński (2005, p. 32).

everything and sets goals for everything. The main Hegelian principle is: "what is rational is real, and what is real is rational".

Thus Hegelian rationality resembles Heraclite's rationality from the point of view of the unity of Logos-Reason and reality which leads to contradictions. Such rationality has a radical nature of processual variability.

In the XX c., classical rationalism[5] was the object of criticism, particularly in continental Europe. Neo-Kantists called its representatives dogmatists. Criticism was also directed against Hegelian forms of rationality, i.e. existentialism and Marxism. The representatives of life philosophy (e.g. W. Dilthey), existential irrationalism (J. P. Sartre, G. Marcel, S. Kierkegaard), hermeneutics (H. G. Gadamer, M. Heidegger) and so called "dialoguers" (E. Lévinas, F. Rosenzweig) were especially loud in their criticism. Radical contesters of post-modernism (J. Derrida, J. F. Lyotard) and post-Marxism (J. Habermas, T. W. Adorno, H. Marcuse) were also influential among the criticizers of classical rationalism. In the XX c., an important role was played in analytical philosophy and science by representatives of rational criticism and logical anti-irrationalism. K. R. Popper (1935) played a leading role in the first group, which opposed various forms of irrationalism (including classical rationalism with its faith in Reason). He did not look for unshakable sources or principles of cognizance (in the sense of Descartes and his followers), but focused his efforts on discovering rules for eliminating errors when acquiring knowledge. Therefore, the criterion of rationality is not the justification of cognitive findings, but seeking falsifiers of the findings. This kind of rationalism is strongly methodological and strictly connected with the modern understanding of what science is. Both H. Albert and J. W. Watkins link their own research to Popper's results. Besides the critical rationalism of Popper, Watkins and Albert, sceptic rationalism was formulated by W. V. O. Quine (1951). The main elements of this approach question the dichotomy of analytical and synthetic truths from the point of view of neo-pragmatism and the possibility of reducing synthetic sentences to sentences of observational protocol. He rejects the coherency theory of truth based on the irrational rationalism of the existence of only one logically coherent distribution of confirmations and negations in an infinite set of possible judgmental propositions without any conditions concerning observations and experiments. The Polish substream of analytical philosophy, called logical anti-irrationalism, stressed an anti-metaphysical attitude, the linguistic part of the clarity of reasoning and strong grounding using advanced semiotic-logical methods. Representatives of this approach (e.g. K. Twardowski, J. Lukasiewicz, A. Tarski, K. Ajdukiewicz, S. Leśniewski,

[5] See B. Paź (2007).

T. Czeżowski, T. Kubiński, A. Wiśniewski, J. J. Jadacki, J. Perzanowski) used forms of conceptual, logical, and methodological rationality. Summing up, it needs to be said that these metaphysical, methodological, conceptual, cognitive and linguistic rationalities discussed above are not the only ones. The axiological rationality of ethical or social norms, values and principles is well known and discussed. In human activities, it is recognizable rationality in action.

For the purposes of this chapter, it is also important to consider the rationality of human expectations.

5.3. Rational expectations and rational predictions

The meanings of rationality discussed in Section 2 throw light on sources of rationality, as well as its forms and criteria. These sources can be found in expector(s), expectatis(es) or the interaction(s) between them [including intertemporal, intersubjectual, interobjectual interaction(s)]. In recent times, mankind has developed a group of physical objects termed artificial-intelligence that can and do serve as technically programmed sensors of levels and changes in expectatis(es) resulting from rational rules and behave according to algorithmic rules of rationality. In a sense, they are also "expector(s)", although they are not equipped with emotional or human-type sensual systems connected to a human brain. For human beings, an expectation of something is a state of mind (currently, many would say a state of his brain) that corresponds to an imaginal projection of a future state or a trajectory of states or image(s) of expectatis(es). If an individual is convinced as to the projectional image, then it will be called his/her belief about the future expectatis(es). A projectional image will be called rational if a man-expector is rational. However, a man is rational if he respects metaphysical, methodological, linguistic, cognitive, conceptual and/or axiological system of rational rules to determine his actions. It would be a great enticement to expect (to believe, hope or not wonder) that it would be easy to find a human who follows such rules of rationality for a long period of his life, but probably only a very small number of humans consistently respect such rules.

On one hand, there are professions which enforce the use of such rules of rationality to determine actions, e.g. surgery, bomb disposal, the fire service. It is said that any representative of such professions who does not respect the appropriate rules which minimize the chance of not realizing a particular task is behaving irrationally. Even the selection of candidates for these professions should be rationally designed and carried out. However, there are also professions where the rules determining actions may be much more relaxed, if not almost irrational. These include the arts, where emotional stress, in particular exciting life histories of the artist or accidental factors, decide about the final effect of an artistic work. The same arguments are valid for the actions of leaders of mystic religions or the activities of some social workers.

The professional aims of an artist are often unclear, fuzzy. Therefore, it is difficult to judge whether an artist has reached his goal in a rational or optimal way from the point of view of the aesthetic value of the final form of his creation. Perhaps, if the time at which work is completed is the only criterion, we can judge his actions according to time-efficiency rules. Thus, from the point of view of an economic observer or merchant, his behavior will be judged to be irrational or not completely rational if the time taken was too long. Certainly, saying to him – that time is money – and that by failing to use this principle he will become economically irrational may not necessarily convince him that this is really the case. The same argument also applies to mystics, missionaries and genuine social workers.

This means that the degree of usefulness of rational expectations depends on the profession performed by the expector(s). Each profession requires general and specific skills. The use of both of these kinds of skills requires the use of rational expectations regarding how the environment will change in the future and how this will influence expectatis(es) via expectors' actions, as well as the rationality of choosing a given form of expectations by expectors(s) from a given or any profession. The highest demand for the use of rational expectations is faced by professional expectors. These are expectors of the following fields of expectatises: demographics, economics, weather, political events, earthquakes, climate change, social uprisings, floods, droughts, wars, scientific laboratory experiments, engineering designs, applied and theoretical models and theories in mathematics, logic, physics, chemistry, biology, astrophysics, biophysics, biochemistry, physical chemistry, mathematical economics, mathematical psychology, mathematical sociology and medicine.

Teachers are another influential group that should use knowledge regarding the rational expectations of pupils and students. In addition, we have politicians, people with clear political and economic power. They should have access to the best knowledge or the best advisers who know not only the actual state of the economy, social ties and tensions, but also the rational expectations of economic and social agents according to professional, social, demographic or economic group, both within one's own country and culturally influential countries.

It is felt by many observers that at present in many developed countries the most influential propagators of rationality could be or are the following social groups: parents, grandparents, teachers, journalists, politicians, policemen and informal leaders of all ages, professions and non-professional interests. It remains an open question as to what percentage of such individuals really care about always being rational in their actions and in deriving their expectations? Or how many of them really care about consistent behavior or deriving their expectations within a framework of a system which is perfectly consistent at least with respect to their experience or the experience of their neighbors, friends or enemies?

Note 5.2. From the point of view of the effect on future trends in expectations, it is especially important to find the distribution of the numbers of people belonging to the following groups: homo astrictus, homo astutus, homo avidus, homo frugis, homo faber, homo ignarus, homo ignavus, homo igneus, homo improbus, homo ludens, homo stultis. Once this is done, it is necessary to find suitable weights expressing their influence on the level of expectations of society for a chosen expectatis.

It turns out, however, that the semantic sense of the words "expectations" and "rational" leads us to many other complex assertions. Some of them are as follows:

- a1) If we assume that "expectation" means an expector's feeling that something (an expectatis) is definitely true, and "rational" means an expector's feeling that an expectatis is true, then the category of rational expectations corresponds precisely to <u>true</u> expectatises. Thus without giving a definition of truth or criteria for its verification, we cannot provide any clear analysis of the meaning and usefulness of this category, in particular with regard to human activities.
- a2) Assuming that "expectation" means an expector's feeling that an expectatis is good for him or others, and "rational" means an expector's or others' feeling that an expectatis is true, then "rational expectations" means that an expector feels that an expectatis is good (also possibly in an axiological sense) and true. Here both "good" and "true" need further explanation.
- a3) Assuming that "expectation" means that an expector has a mental image of an expectatis and "rational" means that the expector ob-

tained this image using a scientific methodology, then rational expectation means a scientifically obtained mental image of an expectatis. In the case of an expector being a computer (with artificial intelligence) linked to robots, such a mental image would mean the image created within the computer processor installed inside the robot or linked to the robot's processor.

a4) Assuming that expectation means that an expector has the hope that an expectatis will turn out to be true, then a rational expectation means that this expectatis will take place if this hope is rationally justified.

a5) Assuming that "expectation" means that an expector is looking (or awaiting) for an expectatis to happen, and "rational" means "scientifically justifiable", then "rational" expectation means a scientifically derivable and justifiable anticipation that the expectatis will occur.

a6) Assuming that "expectation" means an expector's assumption that an expectatis is true and "rational" means that the assumption is scientifically probable, then by "rational expectation" we understand a justifiable, scientifically probable assumption about the assumed expectatis.

a7) Assuming that "expectation" means an expector's notion about the concrete form of an expectatis and "rational" means respecting the rules of axiological rationality, then "rational expectation" means an expector's axiologically coherent notion about an expectatis.

There are other possible meanings of expectation, such as imagination, conception, estimation, opinion, judgement, proposition, presumption, conjecture, guess or supposition. These have similar, but not identical, meanings. Hence, we omit the formulation of further meanings of the terms "rational" and "rational expectations".

It is worthwhile to underline that in scientific studies it is convenient to use the understandings (a3), (a5), (a6) and (a1) after describing which definition of truth is used.

Applying the ancient greek term $\delta o\xi\alpha$ which meant proposition, opinion, notion, judgement, conviction and idea, together with the fact that often $\delta o\xi\alpha$ was used as a synonym of ύπόληψις in e.g. (a6), we obtain a more flexible formulation of the concept of rational expectation.

The concepts of "mathematical expectation", used in probability calculus, statistics, econometrics, biometrics and technometrics, "prognosis", "prognostication", "forecast" and "forecasting" are various types of expectation used in science. If they were or are prepared by an expector using rational (i.e. scientific) rules, then we will use the term "scientific prognosis" instead of "rational expectation".

It is easily seen that the term "expectation" is closely related to the terms "prediction", "forecast" and "prognosis" used in the appropriate literature.

By "prediction", we understand[6] something that is going to happen according to somebody's pronouncement or act of saying what is going to happen.

By the term "forecast", we understand a statement of what is likely to happen in the future, based on the information possessed. The term "prognosis" is understood as a judgement about the future based on known information.

The words "forecast" and "prognosis" correspond to specific pronouncements or opinions that are likely or unlikely and take the forms of statements or judgements based on experience or models for forecasting. Such statements or judgments should be clearly differentiated from something that is going to happen or the act of pronouncing what is going to happen, since an act itself is different from the form (judgment or statement) that it takes. This means that the words "forecast and prognosis" are almost synonyms, but they have a different meaning to the word "prediction", which embraces both the expector's pronouncement of what will happen and the something or the meaning of something that will happen. Use of the word "something" closely corresponds to the meaning of expectatis used in Section 1, where the term "expectation" was analyzed. Thus, the term "expectation" has common roots with the term "prediction" in the sense that both terms simultaneously link the object of expectation or prediction to the person that has an expectation or makes a prediction. For these reasons, it is also convenient to use the terms "object of expectation or prediction" for the concept of "expectatis" and to sometimes use the terms "rational expectations" and "rational predictions" interchangably.

The makers of rational expectations will be called rational expectors and the makers of rational predictions will be called rational predictors. Let us recall (see §1) that the role of such makers can be played by both human beings (human expectors or predictors) and artificial intelligence robots or automats linked to PC-processors and computer programs enabling

[6] See 1).

the processing of information and derivation of mathematical or mathematical-logical expectations or predictions.

In the case of artificial (virtual) expectors or predictors, the parts of the PC-programme “responsible” for “expectation or prediction” calculations translating mathematical – logical formulas (functions), or the functions themselves (in some fields, e.g. econometrics), may sometimes be called predictors. In our framework, it seems more natural to call them the expector’s or predictor’s PC – a programmed mathematical model.

5.4. Adaptive rational expectations and predictions

Up to now it has been assumed that expectors and predictors were not themselves learning by making expectations and predictions and using updated information about their closer and farther environments. Let us now assume that, due to the influence of endogenous and exogenous factors, expectors and predictors may and do gradually change the way in which they make rational expectations and predictions, in order to achieve more flexible (elastic) and possibly improved expectations and predictions. We exclude the possibility of learning only for the sake of learning without fulfilling the aim of improving expectations and predictions, using at least cognitive and modern methodological rules of rationality. In such a case, the use of adaptive (or adjusted or suited) rational expectations and predictions would mean that expectations would be better suited to changes observed in the expectatises or the actual states of the expector’s or predictor’s environments and their possible interactions. This also means that we assume that both expectors and predictors, as well as expectatises, behave in adaptive ways.

It is known from biology and psychology that all living organisms, even viruses, adapt (adjust, acclimate, accommodate) themselves to changes in their environments, i.e. they learn themselves or learn by doing, in order to survive and reproduce. Humans also adapt themselves to the changing conditions of their environment. In the last century, they have both consciously and unconsciously changed some elements of their environment (e.g. regulating and changing (e.g. in the Soviet Union) the flow of rivers, industrial pollution, creating artificial reservoir systems of water, airports, underground train networks, telecommunication networks, water supply networks, gas pipe networks, sewage networks, explosions of atomic bombs, the Internet, mega-cities, vacation facilities, the spreading of hypermarkets which are now becoming temples celebrating the market-culture).

A wide range of educational services supplied by schools of all kinds – from professional through semi – professional to voluntary organizations help in people's learning processes, but do not replace independent learning as probably the most important part of the process of human adaptation. Adaptive changes of the links between events, phenomena, processes and quantities result from human activities, which are subjects of our interest and were already partially introduced in Section 1.

It is instructive to use here the technical or general language of cybernetics[7] regarding the theory of the actions of automatic adaptation (self-adjusting or self-adapting) systems. In such systems, the process of automatic adaptation consists of changing the system's parameters (i.e. the features of a system or subsystem), where adaptation takes place either in an open or automatic regulation (control) system or in a system of automatic search. An optimizer in an automatic optimizing system may act as an example of this. Using automatic search, it changes the characteristics of the primary steering (control) instrument in such a way that the instrument adapts its functioning in the best possible way and the steered system, which receives the appropriate impulses, also adapts itself into another regime of functioning caused by recent changes in the system's features. There is another theory which represents very similar, if not the same, aims and fields of applications. This is the theory of learning systems, (self – organizing systems, self – tuning systems). A question arises as to whether it is possible at all for a living or technical system to adapt itself without adjusting, accommodating, learning, self-organizing or self-tuning (independently of whether the instruments sensing changes are built into the system or they are outside of it, but sufficiently sensitive to identify changes in the system parameters and linked to the system in order to transmit control signals).

The author shares the view that there is no adaptation without learning, self-learning, self-organizing and self-tuning. Therefore, there is no need to add these words each time we use the term "adaptation" – except when we want to stress a particular feature of adaptation.

It is obvious that the objects being studied may undergo adaptation via external influence (effects), adapt themselves autonomously or by a combination of the two. When using technical language, the need to extract the processes or systems of adaptation appears, due to the difficulties of describing behavior without them. This follows from, among other things, a lack of easy access to appropriate information about the internal and external causes of particular changes in a system's behavior, as well as about

[7] See e.g. A. A. Feldbaum (1966), W. R. Ashby (1958), R. Bellman (1961), R. Bellman and R. Kalaba (1959), L. Couffignal (1963), R. Wagner (1954).

the internal conditioning of its behavior. These causes and effects include appropriate means of steering (control) in the form of the sensual organs of living organisms or of engineering devices fulfilling the role of sensory instruments built into the system being studied. Control sensors keep systems functioning under a regime of optimal control (regulation) – (motion) subject to unknown changes in external or internal physical, chemical and biological conditions. As is well known, sensors may fail and can be destroyed by internal or external causes. In describing such situations in the context of adaptive expectations, we should be prepared to find not only various forms of description, but also advise, as well as predict failure times, the effects of sensors failing and costs of repair.

Note 5.3. In the above discussion, a technical or biological system may be replaced by an economic system, e.g. economy, market, firm, household.

Above, we have proposed some simple answers to the following questions:

- who is expecting what?,
- who is forming rational or adaptive rational expectations, why and to what do we attach expectations?
- what does an adaptation process or system mean?
- how is adaptation related to learning, self-learning, organizing, self-organizing and self-tuning?

It should be recalled that the idea of expectations is strongly connected to the field of psychology. This means that it is hard to imagine that there will be any significant progress in the analysis and modeling of expectations, prediction or optimal decision making without taking into account crucial findings regarding the psychology of emotions and stimulus-reaction.

Models that mathematically describe adaptive expectations should correspond in some degree to expectations formed by people in an adaptive way. Such models may take many mathematical forms defined in deterministic or stochastic ways.

Besides the use of adaptive human expectation in economics, sociology and social psychology, the idea of rational expectations formed by people as decision makers is a very fashionable idea. In the light of the above conclusions, a question arises as to whether the category of rational expectations always embraces the category of adaptive expectations or not?

Another question is whether people's expectations[8] can be irrational if they are formed in an adaptive way? One final question is whether the effects of rational or adaptive rational expectations, as reflected by observed events, phenomena or quantities, can faithfully reflect the structural pattern of the original expectations? We shall try to answer these questions in the following chapter.

When discussing notions of links between expectation(s), rationality and rational expectation(s), it must be remembered that from an ontological and epistemological point of view – rationality, together with irrationalism, belongs to the broad stream of philosophical idealism.

We recall that irrationalism includes empirism, fideism, extatic intuitionism, voluntarism and emotivism. Rationalism is in opposition to these forms of irrationalism and, in particular, to sophism and scepticism, shools of thought initiated by Pyrron, as well as the relativism of cognizance. Representatives of rationalism assumed and have been convinced that certainty and truthful episteme (knowledge) has a reliable, autarchic source, which is independent of experience. The source is called *νοῦς*, *λογος* or *διανοια* in Greek, Reason in English, and ratio or cogito in Latin. Pythagoras, Parmenides and Plato were radical rationalists. Critical and empirical rationalists (such as Descartes, Kant, the neo-positivists, K. R. Popper and his followers) belong to a more contemporaneous group of rationalists. In rationalism, the objects of analysis are ideas and concepts. The methods of analysis it proposes depend on the status of ideas (whether they are inborn, discovered, or created).

The methods broadly discussed above can be categorized as either being noetic-eidetic (which penetrate generic ideas and order the world of ideas) or constitutive (constructive-speculative), ideas which form a basis for various systemic ontological visions of the world.

Such authors as R. Descartes, B. Spinosa, G. W. Leibniz, J. G. Fichte, B. Russel and E. Husserl should be mentioned here. Their viewpoints are ontologically static, whose bases (archetypic ideas) are general, necessary, invariant and non-contradictory. When ideas come into being in the empirical world, they fulfill necessary ontological or historical rules (see, e.g., G. W. F. Hegel, A. Comte, C. Marx, A. N. Whitehead). Unfortunately, some parts of Hegel's and Marx's texts were inspirations of the so called modernistic ideological movement. In the XX c., some representatives of this movement used this ideological justification in discussions promoting communism, fascism and nazism.

[8] See, e.g. P. Fisher (1992), H. Blomqvist (1989), G. Shafer (1976), R. G. Miltenberger (2004), B. F. Skinner (1953), R. G. Tharp et al. (1969).

Irrationalism is an important school of idealism. According to this stream of thought, Reason is neither a truthful source of knowledge about the world, nor is it the last authority regarding the evaluation of knowledge, because it is not autonomic or experience specific. Some critics of irrationalism say that, for example, sensualists ignore the existence of genetic links between Reason and the senses, and intuitionists include unclear presumptions or suppositions as relevant hypotheses for studies. The main counterargument against irrationalists is that they formulate judgements which often lead to internal contradictions, are in disagreement with natural experience and release humans from the responsibility for their decisions and deeds. It is an open question as to which economic theories and rational expectations based on them have roots, at least partially, in irrationalism and what is the consequence of this with respect to the explanatory and predictive power of these theories.

Bibliography

Ashby, W. R. (1958). *An introduction to cybernetics.* Chapman London.

Bellman, R. (1961). *Adaptive control processes.* PUP Princeton NJ.

Bellman, R., Kalaba, R. (1959). A mathematical theory of adaptive control processes. *Proc. Nat. Ac. Sci*, USA Vol. 45, pp. 1288–1290.

Blomqvist, H. (1989). The “rationality” of expectations. *J. Ec. Psychology*, Vol. 10, pp. 275–299.

Couffignal, L. (1963). *La Cybernétique.* PUF Paris.

Feldbaum, A. A. (1966). *Osnovy teoryi optimalnykh avtomaticheskikh system.* Naukha Moscva.

Fisher, P. (1992). *Rational expectations in macroeconomic models.* Kluwer Ac. Publ. Dordrecht NH.

Grabman, M. (1909–1911). *Die Geschichte der scholastischen Methode I, II.* Frankfurt.

Kant, I. (1781, 1787). *Kritik der reinen Vernunft.* Riga.

Longman Dictionary of Contemporary English (1995). London G. L. Harlow.

Miltenberger, R.G. (2004). *Behavior modification.* Wadsworth / Thompson Belmont CA.

Ogryzko-Wiewiórkowski, H. (1986). *Problem racjonalności wyboru społecznego.* Lublin.

Paź, B. (2007). *Powszechna encyklopedia filozofii.* PTTA, Lublin.

Popper, K. R. (1934). *Logik der Forschung.* Springer, Berlin.

Quine, W. V. O. (1951). Two dogmas of empiricism. *Ph. Rev.*, Vol. 60, pp. 20–43.

Shafer, G. (1976). *A mathematical theory of evidence.* PUP Princeton, NJ.

Skinner, B. F. (1953). *Science and human behavior.* Mac Millan N.Y.

Tharp, R. G., et al. (1969). *Behavior modification in the natural environment.* Acad. Press N.Y.

Tomczyk, E. (2011). *Oczekiwania w ekonomii.* Oficyna Wydawnicza SGH, Warszawa.

Wagner, R. (1954). *Probleme und Beispicle biologischer Regelungen.* Thieme Stuttgart.

Woleński, J. (2005). *Epistemologia.* Wydawnictwo Naukowe PWN Warszawa.

Chapter 6

On expectations. Part II

Summary

In the previous chapter, the category of expectations was extensively discussed within the broad epistemic background of such categories as rationalism, rationality and irrationality. This discussion covered the concepts of rational expectations and rational predictions at a very general level of gnoseological considerations. Here, our attention is focused on the theoretical issues associated with rational expectations and rational predictions in the field of economics. This discussion is presented in a synthetic way using results from the previous chapter.

Keywords: rational economic expectations, rational economic predictions, theories of expectations, theories of rational economic expectations.

6.1. Introduction

Just as there are many theories of rational economic expectations and models corresponding to them, there are many theories of rational economic predictions and corresponding models[1]. The term "theory" has many meanings. According to Aristotle, "$\theta\varepsilon\varpi\varrho\iota\alpha$" means looking at, observing, considering, investigating, enquiring contemplating and inspecting the order of nature through thoughtful speculation that involves appropriately deep, scientific investigation combined with "innate" methods of confirming truth, as well as, the knowledge and science obtained as a result of this process of activities

[1] For an extensive discussion of the category of a model in relation to categories of isomorphism, homomorphism and the principles of modeling see Chapter 3 of this book.

by rationally acting individuals. Its main aim is to find the causes of effects. In modern times, a "theory" is understood either as a system of propositions, judgements and views or a set of ideas, assessments, pronouncements and statements about trends in phenomena studied by scientists. In the natural sciences, a theory is understood either as a system (set) of theorems, a system of theorems logically deduced from axioms, a system of theorems deduced from axioms and definitions, a system of theorems deduced from a set of conceptual definitions and axioms or a system of theorems deduced from definitional axioms, theoretical or empirical hypotheses, and laws.

In mathematical logic[2], a theory is understood as a Language L together with the set of sentences and formulas belonging to L which fulfill semantic and syntactic rules, where true formulas follow from an axiomatic set of formulas.

The author shares the view that theories may have different degrees of applicability in practice. There are theories that, at a given time, are completely abstract and there are theories that are formulated in a way that includes the need for measurement, data processing and analysis. The latter type of theories are sometimes called "empirical theories" or theories that describe empirical objects (see R. Wójcicki (1974) or F. Suppe (1972)). Theories are characterized by their languages, $\{L_r\}$, the sets of their theorems (laws), $\{th_r\}$, their tools for deriving proofs, $\{P_r\}$, their procedures for measurement and diagnosis, $\{M_r\}$, and their systems of empirical objects, $\{\sum_T\}$. Thus a given theory T empirically oriented via M_T, $\sum_T$, can be defined by

$$T \equiv \left(L_T,\ th_T,\ P_T,\ M_T,\ \sum T\right). \tag{6.1}$$

There exist questions regarding the following: distinguishing between such theories, their identification, validity, confirmation, universality and invariance in time or space or both, as well as the problem of choosing the elements of (6.1).

Even if all these problems had been solved, there are plenty of complex problems regarding the choice of sensible models of these theories. The general issues of model selection in relation to the principles of modeling were broadly discussed in W. Milo et al. (2010). The complexity of theories and model selection will be illustrated in Section 2 on the basis of non-adaptive rational expectations, and in Section 3 on the basis of adaptive rational expectations. Section 4 presents chosen problems regarding the analysis of theories and models of rational economic prediction.

[2] See, e.g. R. C. Lyndon (1966) or Chapter 4 of this book.

6.2. Non-adaptive rational economic expectations: theories and models

There are plenty of articles and books devoted to the description and analysis of non-adaptive rational economic expectations (abbreviated to NAREE). Some of them will be mentioned here in more detail. Others can be found, e.g. in the literature referenced at the end of this chapter. We begin with the historical remarks of J. M. Keynes (1936), who wrote in his General Theory that under great uncertainty economic agents are unable or reluctant to form expectations about future economic or political events.

Having personal experience as a practitioner, he notes the following[3] about expectations:

" ...the facts of the existing situation enter, in a sense, disproportionately into the formation of our long-term expectations; our usual practice being to take the existing situation, and to project it into the future, modified only to the extent that we have more or less definite reasons for expecting a given change".

Earlier, he advised not to attach great weight to very uncertain matters, but rather to attach appropriate attention to the facts that agents can understand. In a sense, an agent who takes such advice when taking his actions is rational in relation to the life-actional criterion of rationality, and at least partially rational in the sense of respecting the rule of cognitive rationality.

What theory of expectations is presented in Chapters 5 & 12 of Keynes' book (1936)? The theory given there is intuitive. It consists of the following assumptions:

ak1) in a given economy, there are the following types of expectors (which may overlap): entrepreneurs (producers, traders, investors), production firms, financial firms, Stock Exchange (SE) brokers, service firms, consumers, buyers and sellers;

ak2) in a given economy, there are the following types of expectatises (objects of expectations): future prices of consumer goods, capital goods (e.g. physical, financial, money, stocks, bonds, options) and human capital goods (in today's terminology); income from future sales obtained by entrepreneurs; a firm's future level of and demand for labor; future purchases of physical and financial capital goods; short, and long term expectations regarding changes; expected income from physical, financial and monetary capital goods; future liquidity of financial, monetary and SE-markets; liquidity of consumption goods markets; expected yield and income generation from these markets; expected demand for consumption goods, future

[3] J. M. Keynes (1936, p. 148).

changes in the structure of the markets for physical, financial, human and social capital; future changes in consumer tastes; expectations regarding the marginal efficiency of different kinds of capital; future confidence levels of entrepreneurs and participants in the financial markets; future share of the number of professional investors in the population of investors;

ak3) all production (output) is for the purpose of ultimately satisfying consumers;

ak4) each entrepreneur (producer or investor) has no choice but to be guided by his best expectations about consumer (effective market) demand when purchasing or producing demanded products, i.e. this time period may be longer than than the period over which the costs of production are incurred; the existence of such a period of positive length implies the need for producers to use expectations;

ak5) entrepreneurs' expectations about the future prices of "finished" output on the market are short-term, but their expectations about future returns from purchasing elements of capital equipment in the form of "finished" products are long – term expectations;

ak6) a mere change in an expectation is capable of producing cyclical movements in the levels of production, employment, prices and profits;

ak7) both short and long-term expectations are subject to revision. Revisions of short-term expectations are gradually and continuously carried out by producers in the light of the production results achieved; in the case of durable goods and financial goods, a producer's short-term expectations are based on the current long-term expectations of the investor, which are liable to sudden revisions and can be responsible for sudden changes in stock exchange prices, rates of returns and the level of business confidence;

ak8) producers base their expectations on the assumption that the most recently realized economic results will continue (inertia), except when there are definite reasons for expecting a change in them;

ak9) the daily output of a producer is determined by his short-term expectations about the cost of output at various scales, expectations of the revenue from this output and his current expectations of prospective costs;

ak10) knowledge of the long and medium-term expectations of other market participants will largely determine both the short-term expectations of producers about their future purchases of capital equipment and distributors' demands for products;

ak11) each state of market expectations has its corresponding level of long-period employment, i.e. the steady level of employment attained under the sufficiently lengthy influence of the current state of expectations;

ak12) a change in entrepreneurs' expectations will, in general, induce various daily changes in stock prices, profits and employment; the economic effects of revisions of such expectations occur over a period of time;

ak13) employment depends more on entrepreneurs' expectations about the future than on the present level of sales or production;

ak14) there are no changes in the interest rate; changes in the values of investments are solely due to changes in the expectations of investors about the prospective yields from investments;

ak15) the existing economic state of affairs will continue indefinitely, except in cases where we have specific reasons to expect a change in this state; entrepreneurs assume this, though they do not believe in the fulfillment of this assumption in real life;

ak16) any existing market valuation of an investment, however arrived at, is uniquely correct in relation to our knowledge of the facts influencing the yield of that investment and this valuation will only change in proportion to changes in this knowledge;

ak17) given assumptions (ak15) and (ak16), investor risk purely results from the possibility of news (new information appearing) in the near future;

ak18) a speculator has unlimited command tin $(0, T)$ of access credit or invest money at the market rate of interest and is satisfied with the prospects of investments he plans to make;

ak19) investment markets which are liquidity and gambling oriented, such as stock exchanges (SE), should be constrained by government through appropriately fixed transaction fees and a transfer tax on all transactions aimed at reducing the number of mega speculations on stock exchanges and the risk of financial crisis;

ak20) a merely monetary policy cannot cure a low rate of investment resulting from entrepreneurs having negative expectations;

ak21) government expenditure plans can change entrepreneurs' expectations about future prices on consumption goods markets and financial markets, as well as expectations concerning the marginal efficiency of invested capital, yields on investment goods, employment, production, costs of production, consumption, stocks of physical capital and confidence in the prospects of the financial and consumption goods markets;

ak22) the population structure of households, firms, investors and producers is constant in the short-term, but variable in the long-term;

ak23) the sum of fixed costs and known expected costs of the use of items of capital are constant during the lifetime of these items, although users may revise their expectations of these costs;

ak24) the marginal efficiency of capital depends on current capital income, the expected future income from capital (future yield from capital), expected replacement costs of capital, expected wages and rate of innovation, as well as the expected rate of sales;

ak25) the relation between fluctuations in the marginal efficiency of capital to fluctuations in the interest rate is an important indicator for studies of business cycles;

ak26) entrepreneurs and consumers can freely choose between physical and financial goods;

ak27) the proportion of capital owners that accumulate capital goods for the sake of accumulation itself and not for investment purposes is marginal and does not have an important influence on the functioning of markets and the expectations of other participants in the market;

ak28) the expectations of professional expectors' are not decisively dominant in the formation of the expectations and decisions of non-professional expectors';

ak29) access to important economic and political, current and past information and predictions is available free of charge;

ak30) the role of the economy is to order and coordinate production, the distributions of income and wealth, exchange of goods, circulation of goods between owners and economic units, accumulation of goods, growth in the levels of factors of production, and production itself, as well as to promote culture.

It can be seen from the list of 30 assumptions in Keynes' theory of expectations that the associated theory will be very complex and rich. Some of the assumptions listed above are implicit (i.e. not introduced in Keynes' book (1936)). These include **ak**1, **ak**2, **ak**22 and **ak**26–**ak**30. Others may appear to be rather descriptive propositions for the behavior of market participants appropriate to Keynesian ideas regarding the effects of events, phenomena and facts in individuals' lifetimes (e.g. **ak**4–**ak**10, **ak**12, **ak**13). They also include highly unrealistic assumptions, where Keynes adopts a conventional belief that entrepreneurs adopt assumptions that are implicit in their daily business, although they know that these assumptions are not fulfilled in practice (e.g. **ak**15–**ak**18). The following are assumptions with theoretical foundations: **ak**3, **ak**11, **ak**14, **ak**23–**ak**25). This group of assumptions is related to Keynes' convictions about appropriate economic policies (see **ak**19–**ak**21). These assumptions can also be treated as conclusions or descriptive opinions about real life economic situations. The implicit axioms of Keynes' theory are as follows:

ax1) entrepreneurs, consumers, financial investors and policy makers usually take decisions which are rational in the economic sense based on the information they have (which may be limited or even very limited);

ax2) the share of irrationally taken decisions is not significantly influential.

From assumptions **ak**1–**ak**30 and axioms **ax**1–**ax**2, the following propositions may be formulated. They represent the consequences of the above assumptions and axioms (Keynesian theory) expressed in a synthetic way, which illustrate a number of new features of the behavior of markets. Their form is as follows:

pk1) a firm's, industry's, economy's function of the total value of unit supply is defined by the ratio between the sum of net income from sales and the expected costs of equipment use for a given level of employment to the level of production corresponding to the stable level of employment;

pk2) the estimates of the unexpected losses of a firm applied in cost analysis and predictions depend on the accounting rules describing fixed costs, as well as the assumed and revisable expectations of the costs of equipment use;

pk3) the propensity to invest is strongly dependent on the marginal efficiency of capital (MEC), which, in turn, is a function of the expected income from a unit of capital bought at time t_a, current capital income, the function of expected replacement costs, rate of expected sales from investment in such capital and relative efficiency of capital objects of different ages;

pk4) falling MEC-values, resulting from decreasing variation in the efficiency of capital according to age, can cause a fall in entrepreneurs' profits;

pk5) the prices of capital objects always adapt themselves to changes in the expectations of the future purchasing power of money;

pk6) expectations of an increasing MEC, lead to increases in the prices of capital goods and a decrease in the interest rate which, in turn, decrease the MEC and increase fluctuations in the MEC that may cause business cycle fluctuations in other spheres of the economy;

pk7) the MEC–transmission channel of expectations is stronger than the interest rate channel;

pk8) the ratio of expected fluctuations in the MEC to expected fluctuations in the interest rate is a leading indicator of BC (business cycles);

pk9) the willingness to invest depends on the MEC, as well as the risks of incurring debt and changes in the purchasing power of money;

pk10) changes in the money supply and demand aimed at speculation depend on the influence of operations on the open market, expectations about future changes in the monetary policy of the Central Bank (CB), the liquidity of the financial markets, revisions of expectations, the difference between the current interest rate R and the rate that is safe for making investments, as well as the interest rate R^{eql} equilibrating the labor market;

pk11) if the interest rate fulfills $R > R^{eql}$, then the long–run R^{lr} depends on current monetary policy, the financial markets, expectations and the rules of the monetary policy;

pk12) any trend in R which is considered by market participants as probably permanent or persistent will be persistent;

pk13) the owners of capital K buy and keep K due to the expected income from K;

pk14) the expectations of the yield on new marginal investments grow, not because of the goal of increasing wealth, but because of an expected growth in demand for capital goods;

pk15) savings that do not increase expected yields do not cause growth in investment, but enforce transfers of capital assets;

pk16) positive income from investments with uncertain expectations regarding the investment yield can occur if $R \geq 0$;

pk17) the expected income from capital K can be expressed in non-monetary units;

pk18) if there exist "narrow bottle necks" in an economy, a sudden growth in inflation can occur; but under high unemployment, even a modest growth in aggregate demand (AD) will lead to a growth in employment with a small rise in prices;

pk19) economic booms are evoked by optimistic (or overly optimistic) expectations as to the future earnings from K–goods; economic crises, in turn, are brought about by large falls in the values of MEC and R not increasing;

pk20) the effects on employment of changes in short–or long–term expectations are distributed over a long future period;

pk21) the expectations process is always changing and there is always an overlapping of the results of expectations that complicates assessing the effects of expectations for individuals, firms, the CB and governments;

pk22) at any time, the level of employment depends not merely on the existing state of expectations, but also on previous expectations, which are partly embodied in the present level of capital;

pk23) there is a large overlap between the effects on employment of the sales revenue from recent output and those of the sales revenue expected from current input;

pk24) the expectations as to the prospective yield on an asset are partly based on facts known to various degrees of confidence and partly on forecasts of future events;

pk25) it would be foolish, in forming expectations, to attach great weight to matters which are very uncertain, and it is reasonable to be guided to a considerable degree by facts about which we feel confident;

pk26) the way in which long-term expectations are used as a basis for decisions does not solely depend on the most likely outcome according to our forecast, but also on the confidence in this forecast;

pk27) the rate of investment depends on the MEC, the level of confidence of a businessmen in the future and their interaction;

pk28) estimates of prospective yields on capital are uncertain due to the real life complexity of the links between real and nominal economic variables, expectations as to their future states and their possible interrelationships;

pk29) in the past, the economy was dominated by firms owned and managed by one person or family, which meant that prospective yields (profits) were equally or less important than the power of entrepreneurship. The share of reversible (e.g. stock exchange) investments was insignificant in relation to the economy as a whole. At present, the share of reversible investments is growing, which makes the use of expectations calculus indispensable, in order to reduce the effects of risk and increase the stability of an economic system;

pk30) even small-scale entrepreneurs play decision games on the SE to earn speculative profits from daily news, traces of changing trends in people's and investors' expectations and changing sources of uncertainty on the SE, as well as financial and commodity markets;

pk31) the share of possibly irrationally made investments in stock capital K_s is growing (due to an increase in the share of unprofessional K_s–investors who have no special knowledge and skills in the field), which increases uncertainty in investment markets;

pk32) the day-to-day fluctuations in the profits of existing investments have a growing influence on various investment goods markets;

pk33) sudden fluctuations in opinions are weakly connected with investments, due to the herding effect in the psychological emotions and behavior of investors with limited knowledge, which can lead to rapid changes not only in conventional valuations of capital, but even cause changes in the rules of valuating;

pk34) professional experts and investors are not interested in making revised capital evaluations or buying "to keep", but in being the first to detect the moments at which trends change, i.e. turning points in prices and their expectations, in profits and their expectations, in wages and their expectations, in production and its expectations, in savings and their expectations, and in interest rates and their expectations, placing an accent on recognizing patterns of changes in opinions, which resembles patterns of evaluations in a beauty contest;

pk35) investment on the stock exchange resembles casino gambling and promotes the development of gambling instincts, propensity to gamble, investing quickly on the basis of technical graphical analysis and short-term expectations, and preferring quick positive rates of return from shares in not necessarily productive companies;

pk36) weakening the confidence of either speculative investors or creditors may be a factor of a collapse in equity prices and a big fall in the MEC, but increasing the confidence of both such investors and creditors is necessary in ensuring an economic recovery;

pk37) the USA–investment markets are rooted in the high level of confidence of American investors, especially non–professional ones, in the long-term efficiency of US–banking, the stability of credit systems and the successful organization of "liquid" investment markets; while the English system has higher transactions costs on the SE, diminishing SE-liquidity is a less significant danger to the market;

pk38) a high level of speculation, liquidity and attracting more and more monetary capital decreases investment in physical capital goods and productive services, which in the medium to long–term can have very negative effects on business cycles, employment, production and aggregate demand and supply, as well as stimulating the propensity of households and firms to hoard large amounts of money;

pk39) a large proportion of the positive activities of economic agents follow from spontaneous expectations, as well as emotions of optimism and confidence, rather than on mathematical expectations, whether moral, hedonistic or economic. These emotions are often animalistic and prompt individuals to action rather than inaction;

pk40) the community benefits from enterprising activities, even if they depend on expectations (hopes) supported by the natural instinct of entrepreneurship and rational business calculations;

pk41) economic prosperity and investment prospects depend on rational and irrational expectations, actions stimulated by a socio-political atmosphere congenial to common trends in opinions that determine spontaneous optimism or pessimism, the confidence of investors and consumers, the business and investment climate, and even reactions to the weather;

pk42) human decisions affecting future events and phenomena (personal, economic and political) do not depend on strict calculations of mathematical expectations, but on the innate urge of economic players to take actions and choices which are as good as possible and consistent with their motives, sentiments and assessment of chances;

pk43) the effects of wrong predictions or expectations are weakened by using compound interest in calculations, in long–term contracts for long–term building investments by transferring risk from the investor to the occupier based on the continuity and security of tenure, by state guarantees and monopoly rights to charge defined rates from the use of investments in public utilities, in state investments by not applying the assumption that the mathematical expectations of their yields are at least equal to the current interest rate;

pk44) long–term expectations depend on their short–term changes as distinct from changes in the rate of interest, and influence the rate of investment, but steering changes in the interest rate based merely on monetary policy is likely to be less effective than policy that uses expectations or predictions of the MEC as a means of diminishing future fluctuations in the MEC for different types of capital.

This list of pronouncements and statements rooted in J. M. Keynes' (1936) theory of expectations can be treated either as his opinions on the role and place of expectations in his theory of the short–term behavior of a given economy or as hypotheses subject to empirical confirmation both for the period of the first half of the XX c., as well as for later periods for the UK–economy or other market economies. It is an open question as to which formal models (that may be formulated on the grounds of listed assumptions, axioms and statements) are more or less consistent with empirical statistical data for chosen economies. The author does not know of any fully extended studies in the literature covering all of the listed elements of our reconstruction of Keynes' theory of expectations, although some elements of his theory can be seen in other theories of expectations. It should be stressed, however, that Keynes' attention was focused on the expectations of labor market participants, as well as on participants in the markets for money and security papers. It should be stressed that the expectations of the latter group of market participants are also the subject of H. Working's theory of economic expectations that is presented below.

6.2.1. H. Working's theory of rational economic expectations

Another interesting theory of rational economic expectations was developed by H. Working (1934, 1949, 1958). His theory mainly concerns the commodities and securities markets, especially the behavior and mechanisms of price formation on these markets. We begin with a presentation of his assumptions. They are as follows:

aw1) there exist the following types of expectors: participants of commodities markets (for eggs, potatoes, wheat, cotton, etc.), participants of stock exchange securities markets, futures markets, consumers, investors, employees of local or central government institutions, politicians, market makers, traders, economic advisers of firms, banks, governments, dealers, speculators, buyers, sellers and producers;

aw2) the expectatises covered by this theory are: commodity prices, the normal or equilibrium prices of security papers (such as stocks and futures), price fluctuations, expectations of consumer demand, expectations of supply; the sentiments, beliefs and emotions of optimism and pessimism of market participants, degree of "haggling and bargaining" in the markets, expectations of investor demand, consumers' and investors' incomes, consumers' and investors' expenditures; the utility functions of consumers, producers, dealers, speculators, investors and politicians;

aw3) the problems faced by market institutions and the regulatory role of competition are important, both theoretically and practically;

aw4) all the prices considered are influenced by expectations and are anticipatory (see also F. Taussing (1921));

aw5) supply and demand do not precisely determine market prices (see also F. Taussing (1921));

aw6) A. Marshall's (1890) view of short-period fluctuations around "normal", i.e. equilibrium, market prices is an important part of the price formation process embracing "speculative maneuvers" in futures markets;

aw7) expectations are an important ingredient in the process of demand formation;

aw8) in order to make current and future decisions, consumers must predict prices and their future income;

aw9) a realistic model of an anticipatory market assumes that, at a given moment, supply is fixed, reservation prices determine demand, prices may be set from moment to moment, "short" and forward sales are possible and influence the present spot price;

aw10) the holdings-demand schedule is based on expected consumption demand and opinions about existing supplies;

aw11) the population structure of expectors and decision makers is known;

aw12) the quality of information is sufficiently good to make predictions of market features with precision not far from measurements made in the future;

aw13) traders, producers, consumers, investors do not haggle (negotiate prices);

aw14) traders have various levels of knowledge, skills, experience, intelligence, abilities and different opinions;

aw15) the share of inept traders and their trading is not significant in a market;

aw16) prices are always formed through the medium of the decisions of economic agents based on the information available to traders;

aw17) the markets considered are similar to real markets;

aw18) the number of traders in the market excludes the possibility of the significant influence of particular traders, and traders do not collude;

aw19) the information available is not corrupted by manipulators;

aw20) most of the traders are able, emotionally stable, equipped with pertinent knowledge and skilled in its use, and devoted to the profession of trading;

aw21) price changes have two components: gradual and non-gradual; they can be recognized statistically based on statistical data, both historical and regarding expectations, in particular using the theory of random processes and analysis of correlations.

These assumptions give grounds for the following axioms:

axw1) most market participants behave rationally in most situations;

axw2) by making expectations and predictions, market participants often improve their consumption and investment decisions;

axw3) when professional traders and investors make informational noise, this is quickly detected by others, and in the medium or long-term is non-effective on a large scale;

axw4) political and social events do not destroy market institutions;

axw5) there are no traders who become too influential based on the size of their capital.

The following are statements based on Working's theory of expectations:

sw1) some of the institutional reforms of the markets in the USA have been conceived in error and administered under delusions;

sw2) the statement following from Taussig's theory that fluctuations in prices are usually limited in scale and cannot be explained purely by supply and demand is, in general, correct;

sw3) the statement resulting from Marshall's theory that the normal price is determined by the normal supply and demand is correct if we replace the term normal by periodical average in each appropriate position;

sw4) excluding intermediaries between producers and consumers and assuming exchange to be based on a grand auction merely forces consumers to become speculators and apply the idea of anticipatory prices;

sw5) the common trend of expectations indirectly driving the decisions of market participants strongly depends on the population structure of market participants;

sw6) transfers of ownership between market participants, the quality and breadth of market information and speed of absorption of information determine both price fluctuations and the quality of the decisions of market participants;

sw7) differences between expectations concerning prices, demand, supply, income, profits, market trends and the quality and significance of new information – are important sources of much trading both in notional and real markets;

sw8) a local abundance of information induces traders to offer specialized services and be market "predators", due to the existence of ill-informed and unskillful traders acting as "prey";

sw9) traders who obtain pertinent market information the most quickly anticipate events more skillfully and by realizing quick transactions they obtain quick profits (sometimes losses), and also increase the frequency and scale of price changes;

sw10) according to Working's theory, price changes are in general unpredictable, unless new information leads to expectations that change gradually, or information is not available to certain traders;

sw11) price changes which are too slow generate a small degree of short-term predictability into such price changes and the danger of possibly large changes in prices with a given probability;

sw12) ill-qualified traders more often choose an investment strategy based on swimming with the market trends, but well-qualified traders often choose moves in the opposite direction using expectations and predictions based on models of random processes and the correlations between the prices and supplies of commodities or stock exchange securities, i.e. historical empirical correlations, as well as estimates of the correlations between expected prices and supplies.

Summing up our reconstruction of H. Working's theory of economic expectations, it must be said that:

- both Working's and Keynes' theories are qualitative and descriptive, rather than aimed at uncovering economic laws;

- like Keynes' theory, Working's theory does not clearly introduce the assumption that economic agents – as creators and bearers of economic expectations – are rational agents, although the assumptions and statements presented above clearly indicate that the majority of agents are rational, and even those who are ill-informed or ill-qualified can learn the basic rules of economic rationality, at least rules learned empirically from real life,

- as in our reconstruction of Keynes' theory, the reconstruction of Working's theory of expectations presented above is far from being perfect. It is discussable whether the choice of the form and content of the assumptions, axioms, and statements are sufficiently distinctive, logically ordered, disjoint, complete and consistent;

- both Keynes' and Working's theories are much broader in content and descriptive in form than the later theory of J. Muth (1961), which is more mathematically oriented in form.

6.2.2. J. Muth's theory of rational economic expectations

The bibliography of J. Muth (1961) shows that his theory of economic expectations – intertwined with price theory – is connected with some concepts used earlier by J. Keynes, H. Working, H. Simon, F. Modigliani and M. Nerlove. His theory is not, however, intuitive (like the theories of Keynes and Working), but a quite formal one. Muth's theory makes the following assumptions:

am1) there exist the following types of expectors: farmers, entrepreneurs, isolated market participants, managers of firms, forecasters, predictors, speculators, households, producers, consumers and traders;

am2) there exist the following types of expectatises: price movements (fluctuations), market structure, demand and supply functions for farms, firms, sectors, market equilibrium, entrepreneurs' predictions, public predictions, income effects on demand, cost effects on the supply from firms, cross-sectional differences in the expectations of market participants, expectations of aggregate prices, inventory levels, speculative and non-speculative demand, speculative inventories, households' accumulation of inventory, losses or gains, aggregate demand (AD), the AD-function, the upper and lower saturation level of inventories, effects of inventory speculation, market stability, elasticity of demand with respect to price, government forecasts, the variability of inputs, outputs, mean income of consumers and speculators, producers' revenue, consumer expenditure, frequency of business cycles, the magnitude and patterns of price autocorrelations, models of prices, demand and supply;

am3) economic expectations regarding future events are essentially the same as predictions resulting from the relevant economic theory and, as such, are rational;

am4) economic expectations as to the values of economic variables may be subject to error;

am5) economic expectations are rational, modelizable and useful;

am6) economic agents, generally, do not waste information;

am7) markets are isolated, market participants are able to speculate and production may not be instantaneous, which influences the choice of methods of analysis by economic agents;

am8) economic expectations depend, in particular, on the structure of an economy and markets;

am9) changes in information systems or the structure of economic systems lead to changes in the way expectations are formed and requires updating the relative importance of various expectations;

am10) dynamic economic theories and their models do not assume sufficient rationality;

am11) expectations of the managers of firms are random functions (or variables) which have a subjective probability distribution of outcomes that is close to the distribution proposed by a relevant economic theory;

am12) the results of entrepreneurial activities cannot be perfectly modeled, and their expectations are different and not perfect;

am13) models of isolated markets at equilibrium are linear and stochastic with normally distributed disturbances, where the theory covers goods markets;

am14) the variables used in the model denote deviations of the values of given variables from the equilibrium values of these variables;

am15) the expected market price at moment t is calculated at moment $(t-1)$, so only one period ahead rational expectations are taken into account;

am16) there is more than one producer and more than one market participant acting as a consumer or buyer, and the expectations of firms and consumers are summable;

am17) the mathematical expectation of the original uncorrelated disturbances, ε, is assumed to be zero, and the aggregate expectation of prices held by firms is equal to the predicted prices according to the economic theory of the behavior of markets;

am18) the market price is expressible as a linear function of the price expectation and vice versa, although in the latter case the expression will be non-linear with respect to the structural parameters;

am19) we only consider models based on the theory of rational expectations that consider price expectations;

am20) based on a basic linear, stochastic model of the equilibrium in a market for a single consumer good, the demand for consumption depends only on the current market price of the good, and its supply, in physical units, only depends on the expected market price, which is unknown, random and usually unpredictable at the time of production decisions, but known at the time a product is purchased;

am21) this basic model may be extended by allowing income effects on demand and effects of the prices of alternatives on supply and the possibility to estimate the original exogenous disturbances $\{\varepsilon\}$, as well as later disturbances $\{U\}$ based on the model;

am22) direct disturbances in supply U_t are expressible as an infinite series of random variables $\{\varepsilon_{k-i}\}$ with weights $\{w_i\}$ enabling correspondence to any given correlogram of U_t;

am23) the market price p_t can be expressed as an infinite series of random variables $\{\varepsilon_{k-i}\}$ with weights $\{w_i\}$ and the expected market price p_t^e is representable as an infinite series of observable prices $\{p_{t-j}\}$ and weights $\{V_j\}$;

am24) the time at which information is accessed/becomes available is important;

am25) speculative manipulations of the inventory level generate profits independently of the storage and transactions costs, as well as the interest rate, and the utility of profits and the expectations of this utility are expressible in terms of Taylor's series about the origin;

am26) the conditional variance of the price is independent of the expected price, and the square of the expected price change is small in relation to this conditional variance;

am27) the AD (aggregate market demand) depends on the population structure of inventory holders (firms, intermediary traders, households), i.e. on the aggregate inventory based on this structure;

am28) a model of such a market with speculative transactions includes an equation describing speculative inventories which depend only on the difference between the expected (at time t for time $t+1$) prices and observed current prices, together with a reformulated equilibrium condition that demand for consumption is satisfied by the current inventories and supply is satisfied by inventories from previous periods;

am29) there exist easily findable conditions for dynamic stability under the basic and extended models;

am30) Ezekiel's, Goodwin's and Nerlove's prediction formulas for the expected price are reasonable short time devices which are helpful in checking the REH (rational expectations hypothesis).

We leave open the question as to whether our reconstruction of Muth's assumptions in his theory of rational expectations is a full description. It could probably be improved in its formulation. It would be interesting to see how coherent this set of assumptions is and whether they are disjoint.

A list of explicit statements resulting from J. Muth's theory of expectations is as follows:

sm1) the Stockholm School's theory of market fluctuations is limited;

sm2) the character of dynamic market processes is typically very sensitive to the way in which expectations are influenced by the processes themselves;

sm3) the structural forms of models of markets are informationally very helpful in analyzing changes in such structures;

sm4) the averages of the expectations of industrial firms are more accurate then the expectations from naive models and as accurate as expectations from elaborate econometric models based on large systems of equations, although there are considerable cross-sectional differences;

sm5) reported expectations generally underestimate the extent of changes that actually take place;

sm6) expectations are rational if they are, as informed predictions of future events, the same as the predictions resulting from the relevant economic theory;

sm7) since economic agents have limited access to information, generally they do not waste it;

sm8) the way expectations are formed depends specifically on the structure of the relevant economic theory or its model;

sm9) adopting Grunberg and Modigliani's concept of public prediction will have no substantial effect on the operation of a market or economy (unless it is based on inside information);

sm10) in the supply equation, disturbances are normally distributed, since exogenous shocks are also normally distributed;

sm11) under the market equilibrium, current market prices are linearly dependent on the expected prices and supply disturbances;

sm12) the mathematical expectations of equilibrium prices based on a model/theory are proportional to prices expected by the firms;

sm13) there exists a constant of proportionality that equalizes the aggregate expectation of firms and the theoretical prediction of the expectation;

sm14) the rationality of economic agents implies that the proposed basic linear equilibrium model based on Muth's theory leads to equality between the theoretical expectations and firms' expectations only in the case of equality between the coefficients of the market prices and expected market prices in the demand and supply equations;

sm15) if the mathematical expectations of the supply disturbances for a given model are non-zero and the corresponding expectations of the equilibrium price are equal to the expected market prices, then the latter expectations are proportional to the expected disturbances, but with the opposite sign;

sm16) the weights $\{W_i\}$ in the infinite series representation of the current price and weights $\{V_j\}$ in the infinite series representation of the expected price are linearly dependent via a triangular linking matrix;

sm17) if all the weights $\{W_i\}$ of the supply equation disturbances $\{U_t\}$ are equal to 1, then the expected price is a geometrically weighted moving average of past market prices based on Nerlove's formula;

sm18) speculative transactions are profitable, but reducing the variance of prices reduces opportunities for speculation;

sm19) a Taylor's series expansion about zero profit enables approximation of the utility function based on profit and the expected utility function in a valid way within a small range of variation in profits;

sm20) based on the assumption that the first derivative of the utility function is positive and the second derivative is negative, the approximation of the level of a speculative inventory is an increasing ratio type function, where the numerator of this function depends on the variance of price forecasts;

sm21) introducing storage costs and a positive interest rate may constrain the aggregate market inventory and cause nonlinearity in the AD-function, such that there exists a fixed upper "saturation" level of inventories, and a lower level which approaches zero;

sm22) given that the level of speculative inventory is proportional to the difference between expected, for moment $t+1$, prices and current prices, the conditions for dynamic stability hold and the exogenous market disturbances are independently distributed, then the expected price is positively correlated with the previous price. When the level of inventories do not play an important role, this correlation should be close to zero;

sm23) if $P_t^e = \lambda_1 P_{t-1}$, then the supply and speculation inventory equations are more restricted and more parameterized. The effects of inventory speculation can be seen by comparing the standard deviations of prices, expected prices and output, as well as by comparing the means of producers' revenues, speculators' revenues and consumer expenditure;

sm24) the effect of inventory speculation on the welfare of consumers and firms is not obvious and is dependent on changes in the coefficient in the inventory equation;

sm25) the rationality principle and rational expectations of entrepreneurs and consumers are unifying roots for many methods of analyzing market economies and such a framework gives grounds for including the effects of systematic biases, incomplete or incorrect information and the imperfect memory of economic agents on the behavior of particular markets;

sm26) empirical applications of the REH (rational expectations hypothesis) are more consistent with historical data for the USA commodity markets than the results of fitting these data using the COBWEB – cycle type models of Schultz, Goodwin and Nerlove; that is to say that the price expected (by firms), P_t^e, is an unbiased consistent prediction of the actual market price;

This list of statements seem to be consequences of the assumptions of Muth's theory of rational expectations. It remains to check whether this is the case and whether the sets of both assumptions and statements are complete, disjoint, logically coherent and empirically valid. It should also be remembered that it is possible to formulate many models based on this theory and that Muth's formalized theory is not the only one.

6.2.3. R. E. Lucas, Jr.'s theory of rational economic expectations

It can be easily seen that despite the fact that J. M. Keynes (1936) presented quite a rich theory of economic expectations, his followers did not try to verify it empirically or discuss its extensions based on their modifications of Keynes' macroeconomic theory. An awakening came in the 70's of the XX c. when two shocks in aggregate supply resulted from oil crises and a period of high inflation, not only in the USA. Great discussion among different streams of economists was evoked by R. E. Lucas (1972, 1973), who described the inefficiency of monetary and fiscal policies aimed at curbing inflation and increasing the growth and employment rates. He underlines the role of the expectations of economic agents within a stochastic equilibrium approach linked to problems related to so called "output–inflation tradeoffs" and the "natural rate of real output". Presenting his theory of rational economic expectations, he explicitly or implicitly assumes the following:

al1) there exist the following type of expectors: market participants, i.e. suppliers of goods and labor, buyers, firms, traders, producers and consumers belonging to two generations, as well as, policy makers;

al2) there exist the following types of expectatises: inflation rates, equilibrium inflation rates, prices on two markets, growth rates of nominal and real output, public psychological expectations, growth in the money supply, relation between the inflation rates in two markets, long–run neutrality of money, intergenerational transfer of money, pre– and post–transfer monetary balances, individual demand for future consumption and employment, distribution function for prices and for monetary transfers, demand for goods, economic state, interest rates, real supply shocks, real and nominal shifts in demand, normal aggregate real demand and supply, the cyclical component of market supply, the effects of fooling suppliers and demanders and the degree to which this occurs, current and normal rates of unemployment, relation between the volatility of aggregate demand (AD) and the general index of prices;

al3) economic agents are rational, base their decisions on imperfect information about relative prices which is available to all and behave optimally according to their chosen objectives and economic expectations;

al4) there are scattered, competitive markets, such that the demand for goods and acceptable price changes are unevenly distributed over them;

al5) the equilibrium processes of the commodity, labor and monetary markets are isolated;

al6) the economy behaves stochastically and a lack of price information for some market participants does not exclude them from making rational decisions;

al7) "rigidities" in the short–run supply from suppliers are due to a lack of, or incomplete, information about prices on other markets;

al8) nominal output is given by $Y(t) = \check{Y}(t) \cdot P(t)$, where $\check{Y}$ is real output, P is the general price level, although it should be noted that $P(t)$ is dependent on suppliers' decisions and market demand;

al9) the elasticity of aggregate demand with respect to price is 1;

al10) each $\log -$ level of market supply $Y^s = Y^{s,n} \cdot Y^{s,c}$ is the log-product of the normal level of supply, common to all markets, and a cyclical component of supply which varies according to the type of market;

al11) the "normal" log-component $y^{s,n} \equiv log\left(Y^{s,n}\right)$ follows a linear trend, but the "cyclical" log–component y_z^c varies according to the perceived difference between the present z–market price P_z, the mathematical expectation $E\left(P_t|I_z\right)$ of the general price index P_t conditional on the information I_z and the log–value of y_z^c at lag one, $y_z^c\left(t-1\ \right)$;

al12) the information $I_z\left(t\right)$ covers the history of y^n and y^c of unsatisfied demand and helps to determine a prior, common to all markets, distribution function $F_{P(t)}$ of P_t, denoted by $F_{P,N}\left(\overline{P}\left(t\right),\sigma^2\right)$, $\overline{P}$ (history of y^n, y^c);

al13) the actual z-market price $P_z\left(t\right) = \overline{P}\left(t\right) + \tilde{P}_z\left(t\right)$, where $\tilde{P}_z\left(t\right) \sim N\left(0,\sigma_z^2\right)$, $\tilde{P}_z$ independent of $\overline{P}$, and suppliers observe P_z and $\left\{\overline{P}\left(t-z\right)\right\}$; moreover, suppliers use this decomposition of $P_z\left(t\right)$ to estimate the correct conditional normal distribution of $\overline{P}_t|P_t\left(z\right)$, $\left\{\overline{P}\left(t-\tau\right)\right\}$ with conditional mean $E\left(\overline{P}_t|I_z\left(t\right)\right) = \left(1-\theta\right)P_z\left(t\right) + \theta\overline{P}\left(t-\tau\right)$, $\tau = \sigma_z^2/\left(\sigma^2+\sigma_z^2\right)$, and conditional variance $\theta\sigma^2$;

al14) there are shocks $\triangle y_t^d$ in the nominal log demand, $y_t^d = \breve{y}_t^d + p_t$, $p_t = \ln P_t$, where $\breve{y}_t^d \sim N\left(\delta, \sigma_{y^d}^2\right)$, p_t is not observed, but expected or predicted at time t; although the demand function is unknown to suppliers;

al15) only unanticipated shocks, $\triangle y_t^d$, may cause changes in output, $\triangle y_t^s$, by "fooling" suppliers (according to this model of the theory of expectations, these changes are assumed to be constant);

al16) fluctuations in relative market prices are caused by the stochastic nature of traders' allocation across markets and stochastic changes in the money supply, $\triangle M^s$;

al17) the only source of (imperfect) information on the current state of real and monetary disturbances is through market prices;

al18) the hedging behavior of agents results from the short run non-neutrality of monetary shocks, $\triangle M_t^s$, and real shocks in supply;

al19) equilibrium prices and supplies are possible vector states of the economy numerically fixed as the solution of the dynamic programming problem;

al20) economic agents belong to 2 generations, where the first produces non-storable goods and the second consumes non-storable goods;

al21) fiat money (i.e. currency that is treated as legal tender, but does not have any backing in the form of a physical commodity) is not inheritable by family members, and for a given assignment of persons to markets, it is not possible to switch or communicate between markets, in which trading takes the form of an auction;

al22) the pre-transfer cash balances of individuals, $m^{s,pre}$, are known to all agents, but the post-transfer balances of individuals are unknown, except for the part revealed by the current P_t;

al23) in each period, the monetary transfers, $x(t)$, and agents' allocations to markets, $\theta(t)$, are statistically independent, and their probability densities are continuous and known;

al24) at any moment t, the state of economy is entirely described by the triple $\sigma(t) \equiv (m^{s,pre}(t),\ x(t), \theta(t))$;

al25) transitions of the state of the economy, $\sigma(t) \to \sigma(t+1)$, are given by the probability densities f_X, f_θ and

$$m^{s,post}(t+1) = m^{s,pre}(t) \cdot\ x(t)\,;$$

al26) the utility of the members of the older generation prefer is strictly increasing in consumption, and their $U(m^s) = 0$;

al27) each of the economy's agents select current c and (expected) future c^e consumption to maximize the sum of two strictly concave and continuously twice differentiable known utility functions $U(c, n^{em})$ and $U(c^e)$, which are both increasing in c, c^e;

al28) there is a known, conditional on the level of current prices and the amount of money an agent holds, distribution function F of future prices and inter-generational money transfers, $x(t)$, as well as of the expected income of agents and price level;

al29) the price index is a function of the economy's current state, (m, x, θ), and the equilibrium price index is a continuous, nonnegative function of the state, where this function fulfills the following condition: the marginal cost of acquiring cash by an individual must be equal to the marginal benefit of that individual in terms of the units of expected future utility gained;

al30) allocations of economic resources are made according to k–percent M. Friedman rules, and that the optimality rules take the market and information structure of the economy as a physical datum.

Assumptions (**al**16)–(**al**30) are taken from Lucas (1972).

The following are implicit axioms of Lucas' theory:

axl1) economic agents wish to maximize joint utility;

axl2) the influence of the disutility resulting from the quantity of some goods being in excess is not significant.

From these axioms and assumptions, R. E. Lucas derives the following statements:

sl1) the expectations of individual suppliers about future demand and prices is, in general, different from the aggregate market expectations or the expectations of outside observers;

sl2) the cyclical component of log-supply $y_z^{s,c}$ is linearly related to log-supply in the previous period and the difference between the price index in market z and its mathematical expectation given the information I_z;

sl3) the log-supply function for market z is a linear function of $y^{s,n}$, $y_z^{s,c}(t-1)$ and the difference $P_z(t) - \overline{P}_t$ and is summable with respect to z;

sl4) the equilibrium price index is a linear function of current and lagged demands, as well as lagged supplies and the current normal log-supply in the markets; the solution is expressible as a function of nominal demand;

sl5) the expectation of the general price level $E(P_t)$ is a linear function of lagged demands, lagged supplies, current normal log-supply and the expectation $E(P_o)$;

sl6) the economy's log-supply $\hat{y}^s(t)$ is a function of the change in log-demand, $\triangle y^d(t)$, as well as $y^s(t-1)$ and $y^{s,n}(t)$;

sl7) the function $\hat{y}^{s,c}(t)$ depends on $\triangle y^d(t)$ and $y^{s,c}(t-1)$; the function $\triangle \hat{P}_t$ describing inflation depends on $\triangle y^d(t)$, $\triangle y^d(t-1)$ and $\triangle y^{s,c}(t-1)$;

sl8) the lag pattern of the equilibrium solution may produce periods of simultaneous inflation and below average real output; these periods arise due to shifts in supply which result from changes in demand not being perceived, but not from autonomous changes in the cost structures of suppliers;

sl9) these equations indicate the existence of a natural rate of output; changes in the average rate of nominal income growth will have no effect on average real output;

sl10) unanticipated shifts in demand have effects on outputs resulting from suppliers being "fooled";

sl11) the prediction that the average deviation of output from the trend does not depend on demand policy is not testable, since this average is zero by definition;

sl12) empirical results for the USA based on the period 1953–67 show a short-term tradeoff between inflation and real output which is not seen in Argentina; the higher the variance of average prices, the less favorable is the observed tradeoff and the higher the variance of demand, the more unfavorable are the terms of the Phillips' tradeoff between inflation and unemployment;

sl13) excluding "monetary illusions" and assuming that all prices are market clearing, all agents behave optimally according to their objectives and expectations, and there exists a systematic short-run relation between inflation according to nominal prices, real output and changes in the money supply;

sl14) in the long-run, changes in the money supply are neutral with respect to inflation, and the real output and money supply are independent of the nominal output, interest rate and expectations of the interest rate;

sl15) the approach of dynamic stochastic programming is more suitable for analyzing the relation between information and expectations than the approach of adaptive rational expectations;

sl16) the utility functions based on current and expected consumption are summable and it is possible to derive the current consumption, production and income that maximizes the utility function based on current and future consumption;

sl17) the price equilibrium function, a solution based on a model of equilibrium, is monotonic with respect to the ratio between the intergenerational transfer of money and market allocation variable, x/θ, in addition, the product representation with respect to this ratio is continuously differentiable and unique; this representation can be simplified if $\theta = 1$ or $x = 1$; more particular forms of solution and properties can be derived by analyzing the density function $f(z, \theta)$ obtained from the density functions $f(x)$ and $g(\theta)$;

sl18) periods where $x_t > \overline{x}$ are good if $\check{y}_t > \overline{\check{y}}$;

sl19) if a monetary rule prescribing ξ-percent growth in the money supply is followed by markets participants, the competitive allocation rule $[\overline{c}(\theta), \overline{n}(\theta), \overline{c}^{\cdot}(\theta)]$ will be Pareto optimal with respect to the rule $[c(\theta), n(\theta), c^{\cdot}(\theta)]$; the merging of markets improves allocations;

This reconstruction of R. E. Lucas' (1972, 1973) theory of rational economic expectations is probably not precise, due to his specific style of writing, and unclear mathematical notation. It should be stressed that the main paper, Lucas (1972 JET), broadly considered models of expectations where the expectors and expectatises are objects which belong to an extremely abstract, theoretical economy similar to the abstract economy considered by P. A. Samuelson (1958).

6.2.4. T. J. Sargent's theory of rational economic expectations

T. J. Sargent (1973, 1977, 1986, 1993) presented a less intuitive theory than those of J. M. Keynes and H. Working, but formally more coherent than J. Muth's or R. E. Lucas' theory of rational economic expectations. His theory makes the following assumptions:

as1) the following types of expectors exist: market participants, suppliers of labor and goods, owners of bonds and equities, monetary policy advisers, decision makers in central banks, consumers;

as2) the following expectatises exist: unemployment rate, general price level, inflation rate, interest rate, real interest rate, money supply, M^s, and demand, M^d, optimal money supply, prices, AD (aggregate demand), AS (aggregate supply), production capacity, unexpected part of price changes, CB's feedback rule for setting the interest rate, CB's feedback rule for setting the money supply, distribution functions of the output, money supply and demand, distributions of shocks in AD, AS, M^s, M^d;

as3) each person's behavior can be described as the outcome of maximizing an objective function subject to perceived constraints, so behavior is rational;

as4) the constraints perceived by individuals in the system are mutually consistent, i.e. individuals' beliefs about others' decisions in a given economy are mutually consistent;

as5) there is a large number of identical firms which treat price p and output $\overline{y}$ of the "average" firm in the industry as given;

as6) in the static case, producers maximize profits at real level of output $\check{y}^*$, the price level p equals the unit marginal cost $c'(\check{y}) = p$; i.e.
$\check{y}^* = \text{argmax}_{\check{y}} \{p \cdot \check{y} - c(\check{y})\}$; where $\check{y} \equiv h(\overline{y}) = \overline{y}$;

as7) in the dynamic case, any agent (e.g. a producer) chooses a plan $\{\check{y}_t^*\}$, such that:

$$\{\check{y}_t^*\} = \text{argmax}_{\check{y}} \left\{ E \sum_{t=0}^{\infty} \beta^t R(\check{y}_t,\ \check{y}_{t-1}, p_t) \,|\check{y}_t = h\left(\check{y}_{t-1}, \overline{y}_{t-1}, U_t\right), p_t = p(\overline{y}_t, U_t) \right\},$$

where $\overline{y}_t = H\left(\overline{y}_{t-1}, U_t\right)$, $\{U_t\}$ is a sequence of i.i.d. random variables, $\beta \in (0,1)$ is the discount factor, E denotes mathematical expectation; so the actual law of motion of the aggregate state $\overline{y}_t$ is given by $\overline{y}_t^* = H^*\left(\overline{y}_{t-1}, U_t\right)$, where the perceived law of motion is $\overline{y}_t = H\left(\overline{y}_{t-1}, U_t\right)$, and $H^* = T(H)$, and so the law defining a rational expectations equilibrium is the fixed point law $H^{*,re} = T(H^{*,re})$;

as8) real demand for money is a linear function of the ratio $P^e_{t,t+1}/P_t$, where $P^e_{t,t+1}$ denotes the expectation of the price level at time $t+1$ made at time t, and given knowledge of μ, defined by $M^s_{t+1} = \mu M^s_t$, and γ, λ, c from $P_t = \gamma M^s_t + \lambda^t c$, the rational expectation $P^e_{t,t+1} = \gamma\mu M^s_t + \lambda^{t+1} c$ is a good estimate of $P^e_{t,t+1}$ in the equation for M^d/P;

as9) there exist relations between the parameters of the real demand, supply and price functions that enable calculation of equilibrium values for prices and money supply, as well as a non-unique equilibrium price in terms of the current nominal supply of money;

as10) in the case of two economies, a constant exchange rate e and the same ratio between expectations for the two currencies, there are well established relations between the parameters of the equations for prices, expectations, real demand for money and equilibrium conditions;

as11) in the corresponding static model, under adaptive rational expectations, the equilibrium values of $\overline{y}^*$ are given by a relaxation mechanism of the form $\overline{y}_i^* = \overline{y}_{i-1}^* + \lambda \cdot \left(\overline{y}_i - \overline{y}_{i-1}^*\right)$, where i denotes the iteration number, $\lambda \in (0, 1)$ and $\overline{y}_i^*$ is the expected value of $\overline{y}$ at iteration i;

as12) in the framework of bounded rationality, it is assumed that the agents take decisions and use methodological rules in the same way as econometricians model a given economy, i..e. they try to eliminate systematic errors in prediction (referred to by T. J. Sargent as "forecast errors");

as13) there is no good theory of how economists behave and learn about the world, although models based on artificial intelligence should be a useful starting point;

as14) the assumption of bounded rationality prevails in both scientific and everyday activities;

as15) models of rational and boundedly rational expectations are a basis for deriving useful decision functions mapping the information held by individuals into decisions and for making abstract experiments which simulate regime changes based on the effects of "pre-reform" and "post-reform" rules on market participants;

as16) public expectations are modeled either using autoregression of a given order with respect to the variables being forecasted or objective mathematical expectations of the formal expressions representing the relations between these variables;

as17) the probability distribution functions of the following are known: real output, the money supply, AD, AS, production capacity, disturbances in output supply, output demand, money demand and supply, exogenous variables, measurement errors for these variables, surprise changes in demands, supplies and prices;

as18) the following are known: parameters describing the relations from (as17), the ad hoc loss or objective function used by policy makers, feedback rules defining interest rates and the money supply, the psychological expectations for one or two periods ahead, the reduced linear form of the structural linear model of the economy;

as19) the authorities may hold the same or richer information than the public, and it is not assumed that there exists an effective systematic rule that affects the unexpected part of the price level or disturbances to the economy;

as20) the information held by the monetary authorities differs from the information held by the public.

From the explicit and implicit assumptions described above, T. J. Sargent draws the following conclusions:

ss1) the REH (rational expectations hypothesis) sufficiently restricts individuals' perceptions about the behavior of other people to make reasonable predictions about expectatises; the bounded version of REH makes this even easier;

ss2) if people's perceptions are inconsistent, then there will exist unexploited utility, possibilities for generating profit within a given economy and room for market disequilibria;

ss3) the requirement that $\overline{\breve{y}} = h\left(\overline{\breve{y}}\right) = \breve{y}$ means that no competitively acting firm has an incentive to deviate from the average output that others adopt;

ss4) the assumptions regarding optimal behavior and the representativeness of each firm induce a mapping $H^* = T\left(H\right)$ from the perceived law of motion $\overline{\breve{y}} = H\left(\overline{\breve{y}}_{t-1}, u_t\right)$ to the actual law H^*; also, knowledge of the equilibrium conditional distribution $F\left(P_{t+1}|\overline{\breve{y}}_t\right)$ is vital when deriving the optimal prices and outputs, P^*_{t+1}, $\breve{y}^*_{t+1}$;

ss5) it is possible to extend Cagan's model of the money market and prices to include the REH and to show the non-uniqueness of equilibrium paths for prices and money demand, to discuss linear models of two economies with fiat money and the resulting prices under a constant unrestricted exchange rate between the two currencies without specifying how agents acquire their beliefs;

ss6) an iterative relaxation algorithm for computing the equilibrium under the REH, $\overline{\breve{y}}^*$, needs some empirical experience in modeling using Cagan and Friedman adjustment (adaptive) expectations, which vary according to the model of prices and money applied;

ss7) J. Muth's REH is optimal if $\breve{y}_t = \breve{y}_{t-1} + \varepsilon_t - \lambda\varepsilon_{t-1}$, where ε is a Martingale process (i.e. the expected value of ε is constant over time) and λ is Friedman-Cagan's parameter describing the strength of people's beliefs;

ss8) it is possible to revise entire expectation-generating functions in response to deviations between actual outcomes and predictions according to $H^*_i = H^*_{i-1} + \lambda\left(T\left(H^*_{i-1}\right) - H^*_{i-1}\right)$;

ss9) assuming that the behavior of economic agents being modeled is similar to the behavior of economists who are modelers leads to either self-referential loops and consequences or feedback type decisions;

ss10) bounded rationality is used to reduce the multiplicity of equilibria computed according to general equilibrium theory using a game-theoretic approach;

ss11) REH-theory is difficult to use in finance under situations corresponding to "a lack of trade" when diversely informed traders extract information from equilibrium prices so efficiently that no trade can occur at any equilibrium or under situations where the policy regimes of various economies change or commitments are fixed;

ss12) under REH, if a government really chooses its targets sequentially (regime change) in time and inconsistently with RE, the response of an associated market would change in ways that depend on the government's motives, when its targets and policies are announced and private agents' beliefs about them;

ss13) REMM (rational expectations monetary models) are good tools to analyze changes between two regimes in macroeconomic policy, e.g. the end of hyperinflation, RE-equilibria under 2 alternative full-commitment monetary-fiscal regimes [i.e. a pre-reform regime with permanently large net-of-interest state budget deficits constantly financed by high growth in the money supply, but less by Government bonds, a regime where inflation is expected to be permanently high and a post-reform regime of a permanently balanced PV (state budget deficit) with no money being printing to finance it and zero-inflation]; hence consistency with REH is not assumed during hyperinflation, but rather that hyperinflation is not too costly;

ss14) regimes based on a "real bills" policy and "quantity theory" show how using REM in "impure" ways that do not fully impose the principles of individual rationality and consistency of expectations provides ample freedom to explain empirical facts which are hard to explain under REH;

ss15) bounded rationality models (BRM) restrict the decision rules of a collection of agents to forms which are easier to encode and update when new information flows in and apply to behavioral schemes of agents based on artificial intelligence;

ss16) under non-bounded REM, when public expectations are mathematical expectations of the variables being forecast which are dependent, among other things, on known policy rules:

a) the probability distribution of real output is independent of the deterministic money supply rule;

b) assuming that the loss function is a quadratic function of the general price level, we obtain the case of an optimal money supply rule that enables equating the expected value of the price level in the next period to a chosen target value;

c) if the CB fixes the interest rate period by period (regardless of how it behaves), the equilibrium price level in an economy is not unique;

ss17) if REM takes the adaptive form of autoregressive schemes, then I. Fisher's and A. Phillips' tradeoffs between output and inflation are deducable and based on (ss16.b) – this case has the form of a well-defined dynamic problem that leads to both a unique optimal deterministic feedback rule for M^s or interest rate R, and to a unique period-by-period equilibrium under a given R;

ss18) switching from AR-expectations to mathematical RE based on T. Sargent's economic model may mean that for some situations Friedman's rule of X-percent money growth would be foolish into one in which such a rule would be the best for the CB;

ss19) the CB may induce fluctuations in real output only through an unexpected change in the general level of prices; however, if the CB and consumers share the same information, under the rationality of expectations, the unexpected part of price changes is independent of the expected part of M^s, so the CB has no decision rule which can affect this unexpected part;

ss20) under a rule that determines the interest rate, economic agents correctly predict the actions of the CB, and the REMM does not give any suggestions as to how and where to anchor the quantity $E_{t-1}P_t$, so there are no indications of how to associate the interest rate with a determined price level;

ss21) if there is an asymmetry of information between the CB and the public, then the tradeoff between real output and the price index is complex, due to, e.g. the complexity of assessing the discrepancy of information between the CB and the public;

ss22) systematic countercyclical macroeconomic policies are fairly robust to changes in the model of the AD schedule and the portfolio balance condition, as well as to switching from an adaptive AR-rational expectations scheme to a non-adaptive RE-scheme;

ss23) the short-run response of the interest rate R and employment to an exogenous shock in endogenous expected inflation affecting AD according to an IS-LM-Fisher-Phillips type model is distributed over time;

ss24) the Phelps-Friedman hypothesis of a natural rate of unemployment is true, and no systematic monetary or fiscal policies can evoke a permanent effect on the U-unemployment rate; together with the REH, this leads to the statement that the real $\check{R}$ is independent of the systematic or expected part of the M^s, so expected inflation Π^{e} only determines the nominal interest rate;

ss25) the adaptive REH of Cagan in the form $P^e_{t,t+1} = \sum_{i=0}^{\infty} \varepsilon_i P_{t-i}$ adapted to the linear equations for the AS, AD(IS) portfolio balance leads to a linear stochastic equation for the nominal interest rate that is useful in verifying Fisher's theory of the real interest rate being independent of the systematic part of M^s;

ss26) the current log-real AS gap, $(\check{y}^s_t - \check{y}^{s,eq})$, is linearly dependent on the error of public expectations in the previous period, $P_t - P^e_{t-1,t}$, and the error in model specification; the current real log-AD-gap is linearly dependent on the actual nominal interest rate corrected by the current error in public expectations, the joint influence of exogenous factors and the specification error; the real money supply depends on the current demand based on national income, the nominal R_t and the specification error; hence, the AS and AD are linked by the price expectations of the public;

ss27) Sargent's model differs from the standard Phillips-Fisher model by replacing fixed-weight, extrapolative or adaptive rational expectations by non-adaptive RE which does not systematically differ from the model's predictions and:

ss28) the rate of output being independent of the systematic parts of M^s and the fiscal policy variables,

ss29) the real $\check{R}$ being independent of the systematic part of M^s,

ss30) no equilibrium price exists under a policy of fixing R,

ss31) the set of lag coefficients for monetary income are variables which depend on, e.g. the M^s -rule , so when a policy changes, this set of lags also change;

ss32) according to Sargent's model (1973 BPEA), equilibrium prices depend on the following: the current and historic values of log M^s, normal production capacity, policy variables and errors in model specification; these equilibrium prices implicitly define the future prices conditional on the systematic part of historical policies;

ss33) the properties of Sargent's model where the AS-gap is related linearly to the "surprise" component of the prediction error for the current price level minus the amount by which the average of price expectations in n future periods is revised due to receiving new information – are similar to the properties of Sargent's original model (1973);

ss34) replacing the capacity gap equation by its counterpart in the form of an AR(q)-gap linear equation leads to a capacity gap equation where the conditional prediction errors of the weighted distributed lags can be represented using an infinite series approximation;

ss35) the surprise, random, part of the inflation rate has a much larger effect on the unemployment rate, U, than the systematic part does, although this effect is often statistically insignificant;

ss36) the relation between U and $\Pi^{\text{suprise}} \equiv \Pi^{\text{sur}}$ is less stable than the relation between U and Π^{e} (in the short-run);

ss37) there is no valid empirical proof that any given model of the hypothesis of a natural level of unemployment is best, so it is reasonable to design employment policies on the basis of the expected value of unemployment rate;

ss38) the equation resulting from the hypothesis of a natural rate of unemployment based on expected and unexpected price changes is empirically valid, but the corresponding equation for wages is empirically rejected;

As in the case of previous reconstructions of theories of rational economic expectations, this reconstruction of Sargent's theory was based on papers cited below. Due to editorial constraints, it is not possible to make deep comparisons of the theories discussed above or to analyze any unclear issues. It is worthwhile to underline that their theoretical and practical usefulness will be confirmed if it can be proved that they are necessary for very good ex ante predictions of real ontic subjects and objects such as, e.g., nominal prices, employment, unemployment, supply of consumer goods, supply of given minerals, wages of employees, profits of owners.

Bibliography

Anderson, P. A. (1979). Rational expectations forecasts from nonrational models. *JME*, pp. 67–80.

Barro, R. J. (1976). Rational expectations and the role of monetary policy. *JME*, pp. 1–33.

Begg, D. K. H. (1982). *The rational expectations revolution in macroeconomics*. Phillip Allan, Oxford.

Cagan, P. (1956). *The monetary dynamics of hyper-inflation*. M. Friedman (ed.), Studies in the quantity theory of money, UCHP, Chicago.

Evans, G. W., Honkapohja, S. (2001). *Learning and expectations in macroeconomics*. Princeton University Press, Princeton.

Fischer, S. (1980). *Rational expectations and economic policy*. UCHP, Chicago.

Fisher, P. (1992). *Rational expectations in macroeconomic models*. Kluwer Academic Publishers, Dordrecht.

Holden, K. et al., (1985). *Expectations: theory and evidence.* Macmillan, London.

Keynes, J. M. (1936). *The general theory of employment, interest, and money.* Macmillan, N.Y.

Lucas, R. (1972). Expectations and the neutrality of money. *JET*, Vol. 4, pp. 103-124.

Lucas, R. E. (1973). Some international evidence on output-inflation tradeoffs. *AER*, Vol. 63, No 3, pp. 326–334.

Lyndon, R. C. (1966). *Notes on logic.* D. van Nostrand, N.Y.

Minford, A. P. L., Peel, D. A. (1983). *Rational expectations and the new macroeconomics.* Martin Robertson, Oxford.

Modigliani, F. et al. (1958). Forecasting uses of anticipatory data on investment and sales. *QJE*, pp. 23-54.

Muth, J. (1961). Rational expectations and the theory of price movements. *Econometrica*, pp. 315–335.

Nerlove, M. (1958). Adaptive expectations and cobweb phenomena. *QJE*, Vol. 73, pp. 227–240.

Sargent, T. J. (1973). Rational expectations, the real rate of interest, and the "natural" rate of unemployment. *Brookings Papers No 2*, Brookings Inst.

Sargent, T. J., Wallace, N. (1975). "Rational" expectations, the optimal monetary instrument, and the optimal money supply rule. *JPE*, Vol. 83, No 2, pp. 241–254.

Sargent, T. J. (1993). *Bounded rationality in macroeconomics.* Clarendon Press, OUP, Oxford.

Simon, H. A. (1958). *The role of expectations in an adaptive or behavioristic model.* Bowman M. (ed.), Expectations, Uncertainty, and Business Behavior, SSRC, N.Y.

Suppe, F. (1972). Theories, their formulations and the operational imperative. *Synthese*, T. 25, pp. 129-164.

Taussing, F. (1921). Is market price determinate? *QJE*, pp. 396–397.

Wójcicki, R. (1974). *Metodologia formalna nauk empirycznych.* Ossolineum, Wroclaw.

Working, H. (1934). A random–difference series for use in the analysis of time series. *JASA*, pp. 11-24.

Working, H. (1949). The investigation of economic expectations. *AER*, Vol. 39, No 3, pp. 150–166.

Working, H. (1958). A theory of anticipatory prices. *AER*, Vol. 48, No 2, pp. 188–199.

Chapter 7

On expectations. Part III

Summary

Part II described reconstructions of theories of rational economic expectations, including the theories of Keynes, Working, Muth, Lucas and Sargent. The present chapter gives a critical assessment of these theories which enables defining extensions and, possibly, syntheses of them, as well as modifying the models of these theories.

Keywords: Keynes' theory of expectations, Working's theory of expectations, Muth's theory of expectations, Lucas' theory of expectations, Sargent's theory of expectations, economic expectations, rational economic expectations, theory of expectations

7.1. Introduction

In Part II, attention was focused on the reconstruction of the economic theories of expectations presented by J. M. Keynes, H. Working, J. Muth, R. Lucas and T. Sargent. The choice of these theories was dictated by their influence on Western economic thought and the economic decisions of important economic agents, especially governments and central banks. It is worthwhile to recall, as was broadly discussed in Part I and partly in Part II, that the category of rational economic expectations covers many dimensions of criteria for rationality, namely metaphysical, methodological, linguistic-semiotic, conceptual, cognitive, axialogical and life-actional. Each economic being may be conscious or unconscious of the direct or indirect influence of the use of these criteria on their expectations, predictions and decisions.

An analysis of the influence of these aspects of rationality on individuals can be carried out thanks to current observations of events by independent observers only in the case of expectations, predictions, decisions that were made in the past. In the case of groups of individuals, this influence can only be roughly enlightened by the use of questionnaires.

Section 2 gives a critical assessment of J. M. Keynes' and H. Working's theories of rational economic expectations. Section 3 presents an assessment of the theories of rational economic expectations given by J. Muth, R. Lucas and T. Sargent.

7.2. Remarks on J. M. Keynes' and H. Working's theories of expectations

It is easily seen that Keynes' theory of economic expectations is one of the least formal ones, but contains rich economic content. It is also relatively easy to find gaps of varying importance in this theory. We list them in the order used in Part II when presenting the assumptions, axioms and propositional sentences of the theory.

The remarks corresponding to assumptions **ak**1–**ak**30 are as follows:

- **ak**1) the list of expectors does not cover: politicians, journalists, artists, religious leaders, etc.;
- **ak**2) the list of expectatises does not directly embrace: future supplies of consumer goods, K_{ph}-physical capital goods, K_h-human capital goods, insurance goods, forward contracts and futures, BC (business cycles), political cycles, timing of stock exchange crises, currency crises, banking crises, financial crises, economic crises, the sentiments and confidence of consumers and investors, unemployment rates in the K_{ph} and K_f goods market, the potential and equilibrium value of GDP and GNP and their growth, market sizes and pressure, the average efficiency of the use of capital goods;
- **ak**3) not all output serves to "ultimately satisfy a consumer"; part of it serves to invest, to save or to hoard goods for speculative reasons, or simply to keep things;
- **ak**4) entrepreneurs have a choice. They can use their expectations not only about the potential number of future consumers and effective demand, but also with regard to people's future demand for investment; they may also order professional predictions of these demands;

ak5) these expectations may be extended to unfinished goods, to future returns from K_f-financial capital goods, K_h-goods and K_{ari}-author rights and intellectual capital goods;

ak6) unfortunately, not only changes in expectations, but also unexpected or expected earth-quakes, draughts, floods, volcano eruptions, wars and revolutions may produce BC-effects;

ak10) these long-run expectations are unknown and impossible to measure precisely; producers can only order or evaluate predictions from professional forecasters;

ak12) expectations may or may not induce changes in: production, inventories, sales, investments, consumption, the marginal and average efficiency of capital used, the level of competition, consumer and investor confidence, attitudes to technical and social progress;

ak13) it is rather the case that employment now and in the near future depends on entrepreneurs' expectations about the "future";

ak14) changes in the values of investments depend not only on the expectations of investors, but also on their predictions and, especially, the decisions based on them;

ak15) it is rather the case that agents often do not believe in the persistence of the present state of things;

ak17) it is not only new information regarding the near future that determines the level of risk of an investment, but also changes in the investor's actual and future competitive position, belief in his abilities to carry out business, and belief in good fortune;

ak18) unlimited access to credit is an abstract assumption and in real life situations is non-fulfilled;

ak19) even when brokers are not physically present at the SE (each investor has an Internet link with a SE-mega-computer), progressive taxes on capital transactions together with safety systems cannot eliminate SE-crises due to the possible activities of computer hackers and the impossibility of always balancing demand and supply;

ak24) the form of capital is not specified; in the case of human, social and intellectual authorship rights, this list of determinants of the marginal efficiency of capital is too restrictive.

A question arises as to whether the listed assumptions were effectively used by Keynes in formulating the listed propositions. It will be possible to answer this question from the following remarks:

pk1) the only assumption that can be indirectly attributed to **pk**1 is **ak**23;

pk2) corresponds to **ak**7 and **ak**23;

pk3) is indirectly related to **ak**9, **ak**14, **ak**20, **ak**26, **ak**29 and directly to **ak**24, **ak**25;
pk4) follows from **pk**3;
pk5) has no direct relations with the listed assumptions;
pk6) follows from **pk**3, **pk**4, and is linked to **ak**6;
pk7) is obvious in the context of **ak**14, **pk**3, **pk**4 and **pk**6;
pk8) contradicts **ak**14 and **ak**25;
pk9) is only indirectly connected with **ak**14;
pk10) has no links to the assumptions listed;
pk11) is related to **ak**20;
pk12) if **ak**14 means that $\triangle R = 0$, then it contradicts **pk**12, but if it means that there are no changes in the trend of R, then **pk**12 follows from **ak**14;
pk13) indirectly corresponds to **ak**24;
pk14) has no direct links to the listed assumptions;
pk15) has no direct relations to the listed assumptions;
pk16) has no direct links; under the second interpretation of **pk**14, it is indirectly linked to **ak**14;
pk17) and **pk**18 have no direct links to the assumptions;
pk19) connects **pk**18 with **ak**25;
pk20) is related to **ak**12 and **ak**13;
pk21) is partly connected with **ak**7, **ak**28 and **ak**29;
pk22) and **pk**23 complement **ak**11, **ak**13, **pk**20 and **pk**23;
pk24) **pk**25, **pk**26, **pk**27 and **pk**28 are not directly linked to any listed assumption;
pk29) this description of the evolution of the market from being dominated by family businesses is based on Keynes' opinion about the state of economic life rather than a conclusion stemming from the listed assumptions;
pk30) – **pk**44) are descriptive pronouncements regarding the possible or factual states of the capital markets, as well as the actions and policies of governments and central banks.

The above remarks on Keynes' theory of expectations show that this theory is not economically or mathematically complete, neither is it logically ideally coherent, and has assumptions, axioms and propositions of theses which are densely and directly connected. Fortunately, these features leave a broad space for possible reformulations, corrections and rectifications that will serve as an admissible basis for defining formal models of new versions of Keynes' original theories.

Similarly to Keynes' theory, H. Working's theory of economic expectations is broad in its economic content and descriptive in form. The following remarks are useful comments on the assumptions **aw**1–**aw**21:

aw1) this list of expectors does not directly cover: journalists, artists, religious leaders, leaders of influential youth movements, etc.;

aw2) this list of expectatises does not directly cover: future supplies of consumer and investment goods, K_h-goods, insurance goods, phases of the BC, political cycles, phases of other (SE, currency, trade, banking sector, financial sector, economic) cycles, unemployment rates in the K_{ph}, K_f-goods market, the effect of inertia and friction on consumer commodity markets, financial markets, the level of entrepreneurship and competition, etc.;

aw7) expectations are not only an important determinant of demand, but also of supply, profits, costs, saving, economic efficiency, market efficiency, etc.;

aw8) not only consumers must take their expectations into account when making decisions, but also investors, politicians, etc.;

aw9) this model of an anticipatory market might be realistic only for SE-market goods;

aw10) such holdings-demand schedules are only valid for participants in the consumption goods market;

aw11) the population structure is only known up to the present day. The future structure is unknown;

aw13) the assumption of no haggling is very strong;

aw14) the assumption of various levels of knowledge and skill is generally a very acceptable assumption. Its consequences may be unexpected depending on the proportion of non-average influential rational or irrational traders;

aw15) the assumption that the proportion of inept traders is small follows from the assumption of group rationality;

aw17) , **aw**18, **aw**19 and **aw**20 are implied by the assumption of group rationality, but not always fulfilled in reality.

aw15)–**aw**20) are linked directly or indirectly with axioms **axw**1–**axw**5, derived from Working's theory.

The following are remarks on the statements of Working's theory of expectations:

sw1), **sw**2) report facts of economic history;

sw3) is a correction of Marshall's statement indirectly linked to the theory of expectations;

sw4) is linked indirectly to **aw**3, **aw**9, **aw**14 and **aw**17;

sw5) is linked directly to **aw**4, **aw**7, **aw**9, **aw**11, **aw**13–**aw**15 and **aw**20;

sw6) is linked to **aw**4, **aw**6, **aw**12, **aw**14, **aw**16 and **aw**19;

sw7) is connected with **aw**1, **aw**4, **aw**7, **aw**8, **aw**10, **aw**12 and **axw**2;

sw8) is connected with **aw**12, **aw**14 and **aw**20;
sw9) is related to **sw**8 and **aw**21;
sw10) is related to **sw**8 and **sw**9;
sw11) is related to **sw**10 and **aw**6;
sw12) is related to **aw**21, **aw**6, **aw**14 and **axw**2.

It can easily be seen which assumptions and axioms are not connected with statements **sw**1 – **sw**12. These are **aw**2, **aw**5, **aw**18, **axw**1 and **axw**3 – **axw**5.

7.3. Remarks on J. Muth's theory of rational expectations

Muth's theory of rational economic expectations is written in a language that uses concepts from probability, statistics, economics and finance. Looking more carefully at the assumptions and axioms presented in part II, it can be seen that there are some links between them and the sentences of the theory. The following remarks are worthwhile for further use:

am1) this list of expectors does not cover: journalists, politicians, trade union members and leaders, artists, religious leaders and priests, professional elites;
am2) this list of expectatises does not include: AS (aggregate supply), AS-functions, AD-shocks, AS-shocks, liquidity (at micro, meso and macro-levels), K_{ph}, K_f, K_h, K_s, K_{ari} – supply and demand (factual and expected), investments in capital, supply of and demand for different categories of money, taxation policy, monetary policy, public debt, household debts, firms' debts, public investments, the level of entrepreneurship and competition, etc.;
am3)–**am**6) these assumptions are consistent with the methodological and cognitive criteria for rationality;
am7) the isolation of markets (see also **am**13) is a theoretical postulate; the assumption of production not being instantaneous is obvious;
am8) markets structure is embedded into economy's structure;
am9) the assumption that changes in information or economic systems lead to changes in the way expectations are derived has both assumptive and predictive consequences;
am10)–**am**12) are more inferential than assumptive;
am11) does not directly assume that the expectations of consumers, investors, etc. are economically rational;

am13), **am**19), **am**21)–**am**23), **am**26), **am**29)–**am**30) contain the formal theory and the assumptions of any model of it;
am20), **am**24)–**am**25), **am**27)–**am**28) deal with the theory of expectations and assumptions specifying models of this theory.

It can also be seen that the degree of coherence of the assumptions is reduced by the disjointness of **am**3–**am**6 from **am**7, **am**10, **am**11, **am**13, as well as, in particular, **am**19 from **am**11 and **am**25. Group coherence is easily seen in the sets of assumptions **am**3–**am**6 and **am**8–**am**9.

The following comments on the list of statements in Muth's theory are useful:

sm1 this judgement on the limits of a particular system of thought is suitable to any theory;
sm2 the sensitivity of processes to the way in which expectations are formed is a propositional judgement linked to **am**8, **am**9;
sm3 is indirectly linked to **am**9;
sm4 is linked directly or indirectly with **am**3–**am**6, **am**8 and **am**11–**am**12;
sm5 that expectations underestimate the changes which actually take place is a factual statement based on empirical observation;
sm6 is essentially just **am**3;
sm7 almost follows directly from **am**6;
sm8 is a mapping of **am**3;
sm9 is not linked to the listed assumptions;
sm10 is not directly linked to any assumption;
sm11 is partly linked to **am**18;
sm12 is not linked to any of **am**1–**am**30;
sm13 this is a fuzzy statement, which hinders making any correspondence;
sm14 is weakly linked to **am**3;
sm15 is not directly linked to any of **am**1–**am**30;
sm16 is linked indirectly to **am**23;
sm17 follows from **sm**16;
sm18 is indirectly linked with **am**28;
sm19, **sm**20 are connected with **am**25;
sm21 is linked to **am**25 and **am**27;
sm22 is linked to **am**28 and **am**29;
sm23', **sm**24' are linked with **am**28;
sm25 is linked to **am**4, **am**5 and **am**12;
sm26 statement based on the analysis of empirical data.

These remarks show that the assumptions **am**7, **am**10, **am**13–**am**17, **am**19–**am**22, **am**24 and **am**26 were not apparently used by J. Muth in presenting the statements of his theory.

7.4. Remarks on R. E. Lucas' and T. Sargent's theories of rational economic expectations

7.4.1. Lucas' theory

Lucas' papers (1972, 1973) testify that his theory of rational economic expectations is focused on the macroeconomic aspects of expectations. The provocative and categorical style in which Lucas wrote, especially his statements that monetary policies are useless in steering an economy and that evaluation of previous econometric policy is the wrong way of doing it, evoked great discussions among economists and econometricians and made Lucas' name well-known. Regarding the assumptions of his theory, it should be said that:

al1) this list of expectors does not include: managers of firms and banks, the elite among capital owners, the culture elite, journalists, educationalists and scientists, practitioners and employees of the medical industry, members of the uniformed services (military, police), black market leaders, show business leaders, religious leaders etc;

al2) this list of expectatises does not include: expected interest rates, expected credit risks, stocks prices, bond prices, option prices, prices of raw materials, energy prices, exchange rates, investments, liquidity, market developments, insolvency, earthquakes, wars, draughts, floods, volcano eruptions, epidemic diseases, level of entrepreneurship, measures of technical progress and expectations of its changes, monetary value of the demand and supply of human and social capital, changes in the structure of financial capital, phases of economic, trade and political cycles, financial crises, currency crises, banking sector crises, etc.;

al3) the lack of an explicit quantifier makes the postulate about the rationality of agents a little fuzzy;

al4) this assumption of the scattered nature of markets could be either an abstract or protocolar-type postulate;

al5) the isolation between these markets is an abstract postulate;

al6) the postulate of the randomness of an economy recognizes that we do not have precise knowledge of how an economy works, so rational decisions without information about prices means that either the role of prices is insignificant when taking decisions or even that rationality corresponds to random chance;

al7) since it is not said how these "rigidities" might manifest themselves and only prices on other markets are considered as key decision variables, this postulate is unclear;

al8) this is a narrow, abstract postulate, which additionally restricts any practical applications of Lucas' theory of expectations, due to its links to statements **al**6–**al**7;

al9) this is not a clear postulate. If it originates from the equation $M \cdot V \equiv P \cdot Q$, then the price elasticity of real $G\check{P}P \equiv Q$, $P \equiv \check{AD}$ aggregate demand is equal to $\frac{P}{Q} \cdot \frac{\partial Q}{\partial P} = -1$; if it comes from the Cassel-Cobb-Douglas production function then, at equilibrium, the price elasticity of real output is also equal to -1; if $P \cdot Q = P_C \cdot Q_C + P_I Q_I$, C–consumption, I–investment, Q_C–real consumption, Q_I–real investment, P, P_C, P_I are the appropriate general price indexes, then the elasticity of real output Q with respect to the general price index P is also equal to -1, yet this is not the case if one computes the elasticity of Q with respect to $CPI \equiv P_C$ or with respect to $IPI \equiv P_I$, when the absolute values of these elasticities are both less than 1;

al10) this is a definition of the normal level of supply and the cyclical component of supply;

al11) these are theoretical postulates about the ex ante prediction of $y^{s,n}$ and $y^{s,c}$;

al12) this is an unclear theoretical postulate about the possibility of linking $I_z(t)$ and the theoretical distribution of prices $F_{P,N}(\cdot)$;

al13) these are theoretical postulates, which require concretization to establish interesting results (even theoretical ones);

al14) if p_t, $\check{y}_t^d$ and y^d are unobserved at time t at macro level by both suppliers and consumers, then, indeed, they can only make expectations and predictions. This postulate seems reasonable. For micro level, particular goods and local markets, this is not the case;

al15) constant changes $\triangle y_t^s$ may mean $\triangle y_t^s =: \triangle y^s$ or $\triangle y_t^s = a_0 + a_1 t$, where a_0, a_1 are fixed parameters; it is unclear which meaning was used;

al16) CB–governors and traders would be surprised that their decisions about $\triangle M^s$ or acting in a given market are described as random decisions; this evokes doubt regarding this postulate;

al17) why is it assumed that statistics on firms and household credit, public debt, interest rates, exchange rates, supply and demand shocks, etc. do not give any information about monetary disturbances?;

al18) hedging behavior also results from the awareness or knowledge of business risk, the propensity to hedge risk and/or awareness that such behavior is economically rational;

al19) this is a methodological postulate regarding the equilibrium state;

al20) why is it important that non-storable goods are consumed only by the second generation of agents and not by the first? what about other goods? what is the share of non-storable goods in the whole output? this seems a very restrictive postulate;

al21) according to this postulate, fiat money cannot be a source of capital accumulation; auction trading is very restrictive in a real economy;

al22) this is a very strange postulate. Without introducing both time and a socio-economic-political system, it is difficult to understand such a switch from an instant of time where there is full open information for all agents in an economic system to an information system which is fully closed to all agents. Was such a state conquered by agents from outside?

al23) is an abstract, statistical postulate about the distribution of transfers and agents' allocations to the market;

al24) this state vector reflects only part of the monetary situation of an economy;

al25) this postulate expresses only how the monetary situation of an economy evolves;

al26) this corresponds to **al**21;

al27) this corresponds to **al**21, together with the assumption of additivity of the utility functions;

al28) implicitly, there is no postulate of the existence of a statistical dependence between expected prices, money transfers, the expected incomes of agents and prices;

al29) current and equilibrium prices are assumed to be functions of only the monetary side of economy, where equilibrium prices additionally satisfy the constraint that marginal costs and benefits must be equal;

al30) the allocation of resources depends only on the k-percent rule (M. Friedman's rule stating that the money supply should be increased by the same percentage each year);

Summarizing these comments, it should be said that the assumptions **al**5, **al**16–**al**18, **al**21–**al**25 and **al**28–**al**30 are directly linked to the monetary theory of economics.

The assumptions **al**2–**al**3, **al**11, **al**13–**al**15, **al**27–**al**29 are directly related to expectations. Hence, 21 assumptions and two axioms are not related to expectations. More comments are needed as far as Lucas' theory of expectations is concerned. They should be devoted to the applicability of the assumptions when reformulating the theory's statements. The statements

sl1, **sl**2, **sl**5, **sl**13–**sl**16 have direct logical relations to Lucas' theory of expectations, i.e. approx. 37% of all the statements. The following remarks on particular statements might be useful for those studying Lucas' theory:

sl1) is indirectly connected with **al**3, **al**11, **al**14 and **al**28;
sl2–**sl**3 are directly connected with **al**11;
sl4–**sl**5 have no direct connections;
sl6) has no direct connections;
sl7) has no direct links, but is indirectly linked to **al**15;
sl8–**sl**9 is directly related to assumptions **al**5, **al**11, **al**19;
sl10) follows from **al**15;
sl11) has no direct links to any of the postulates;
sl12') these are factual statements based on empirical research;
sl13) is partly related to **al**3;
sl14–**sl**15 have no direct relations to postulates;
sl16) is indirectly linked to **al**27;
sl17) is linked to **al**23 and **al**28–**al**29;
sl18) is linked indirectly to **al**23 and **al**27–**al**29;
sl19) is indirectly related to **al**30 and **al**27.

Hence, only the assumptions **al**3, **al**11, **al**15, **al**23, **al**28 and **al**29 were used directly, i.e. 6 assumptions from 30, i.e. 20% of them.

Of these 6 assumptions, only 3 belong to the group of 9 dealing with expectations. Moreover, among the 7 statements directly related to expectations, only statement **sl**2 is connected directly with **al**11 and **sl**13, as well as partly with **al**3. Thus 5 of 7 expectation oriented statements were not logically connected with expectation oriented assumptions. This shows that Lucas' theory of rational expectations is not very strongly connected, logically or in terms of economic theory.

7.4.2. T. J. Sargent's theory

T. Sargent's publications are more carefully written than those of J. Muth or R. Lucas. However, it is still instructive to present some remarks on the assumptions and statements of his theory. We begin with some comments on the assumptions:

as1) this list of expectors does not include: elites (political, military, judicial, medical care, industry, trade, banking, academics, artists, media owners, journalists, managers and owners of firms), investors, consumers, etc.;

as2) this list of expectatises does not cover: the equilibrium unemployment rate, expected unemployment rate, supply of and demand for physical, financial, human and social capital, expected inflation for consumer goods and investment goods, domestic supply and demand, exchange rates (nominal, real, effective, equilibrium), expected exchange rates, expected (nominal, real, equilibrium) interest rates, rates of growth of GDP, real GDP, GNP, real GNP, K^s_{ph}, K^s_f, K^s_h, K^s_s, investments, investment rate, saving rate, productivity of various forms of capital (including labor); the level of entrepreneurship and competition, markets, measures of pressure, etc.;

as3) since no distinction is made between consumption or investment, individual or social behavior and the objective function is undefined, this postulate is very general and abstract;

as4) the consistency of perceived constraints is a very important postulate for Sargent's theory of expectations with far reaching consequences, which are only modestly exposed by the author. He did not specify whether the statement "the constraints perceived by everybody are mutually consistent" means that the constraints are algebraically consistent, ensuring the existence of a solution to the corresponding constrained optimization problem, or that specific real or virtual economic (or socio-economic or socio-economic-political) restrictions occur or are assumed, which are not contradictory in their meaning or intention. These restrictions may take the form of: state legal regulations, the decisions of managers or owners of firms, banking or national guidelines, the financial budgets of economic agents. Treating these perceived constraints as "perceptions/beliefs about others' decisions (that) are mutually consistent" is an overstatement. Firstly, people's expectations about others' decisions and people's perception of the constraints on their own behavior and on the behavior of others are not the same sets of designates. Secondly, mutual consistency of everyone's expectations might mean an expectation commonly shared by everyone regarding all the future decisions of others (consumers, investors, politicians, professionals, etc.). This interpretation seems to be obviously false. A postulate in which "all" is replaced by "some", the relation between "some" and "others" is specified in percentage terms and the list of expectatises is short would be cognitively more pragmatic. In such a case, it will be possible to at least infer the direction of the influence of expectations;

as5) this postulate implies that the owners and managers of many identical firms understand the theory of a representative firm that is treated as a source of expectations. This is an elegant, but purely theoretical, assumption;

as6) this postulate means that producers behave according to the rule that marginal profits are equal to zero, i.e. that they are price takers and so act as if markets are competitive. It is a pity that the author did not mention that the task of maximizing profits is solvable, at least in theory, by specifying the shapes of the demand, cost and profit functions. As, e.g., H. Varian (1984) and H. Brems (1968) show, this is not always the case, even when the shapes of these functions are very simple. Implicitly, there is an assumption that the producer knows these functions and their arguments and thus there is no uncertainty;

as7) the consequences of assuming that producers have different motives (as well as, implicitly, consumers, investors, etc., which strongly distinguishes **as**7 from **as**6) are important. Agents maximize the conditional expected total value of their discounted revenues under formally highly demanding restrictions on the level of activity. Here, randomness play a fundamental role. The concept of rationality obtained in this way is more methodological than cognitive or axial;

as8) this is a model-type postulate about the theoretical measurement of price expectations and their use in the equation describing the real demand for money;

as9) an assumption of a model of monetary theory;

as10) similar to **as**9;

as11) describes a model of adaptive rational expectations that give equilibrium production levels of Cagan-Nerlove type;

as12) an implicit assumption of Muth type expectations;

as13) a methodological postulate of how economic theory is formed; implicitly, this is indirectly linked to Muth's theory of expectations;

as14) the assumption of the prevalence of bounded rationality in day to day life is a reasonable postulate;

as15) see **as**14;

as16) this is a useful operative postulate indicating ways in which expectations can be modeled;

as17) knowledge of these distribution is a highly formally demanding and theoretical assumption;

as18) the assumption that these parameters and the objective functions of agents are known is theoretically elegant, although abstract;

as19) this assumption recognizes the role of surprise changes in prices and the lack of precise knowledge about an economy;

as20) the assumption that the monetary authorities possesses different information to the public is a pragmatic postulate.

It can be easily seen that of these 20 assumptions, 8 (i.e. 40%) are directly involved with expectations, and of these 8, only 4 directly refer to rational expectations. The other 4 of these 8 assumptions implicitly refer to rational expectations. Hence, it implicitly follows that in all 8 postulates the author considers methodological rationality.

We have listed here 35 statements describing Sargent's theory. Among these, 24 (i.e. about 69%) are directly connected with his theory of rational economic expectations. This is a remarkable degree of compactness and logical connection, at least in comparison to the theories of Keynes, Working, Muth and Lucas.

To be more concrete regarding particular statements about the expectation oriented assumptions, we have:

ss1) is directly linked to **as**11 and **as**15;
ss2) is directly related to **as**4;
ss5) has no direct connections;
ss6) is connected with **as**11;
ss7) is connected with **as**11;
ss8) is directly related to **as**7;
ss11–**ss**14 have no direct connections;
ss16–**ss**20 are only indirectly linked to **as**16 and **as**18;
ss22, **ss**24–**ss**27 are indirectly related to **as**15 and **as**18;
ss28) is indirectly related to **as**18;
ss33, **ss**35 have no direct connections.

In the case of statements that are not linked to the concept of expectations, it is worthwhile to mention:

ss3) is linked to **as**5 and **as**6;
ss4) is linked to **as**7;
ss9) is linked to **as**12 and **as**13;
ss10) is linked to **as**12;
ss15) is linked to **as**13;
ss21) is connected with **as**19 and **as**20;
ss23) is indirectly related to **as**17;
ss28) has no direct links;
ss29) is indirectly linked to **as**17 and **as**18;
ss31) has no direct links;
ss32) has no direct links.

Summing up these remarks on the statements of Sargent's theory of expectations, it should be said that:

- only assumptions **as**4, **as**7, **as**11 and **as**15 from the group of 8 expectation oriented assumptions were used directly in formulating the expectation oriented statements of Sargent's theory;
- only assumptions **as**5, **as**6, **as**7, **as**12, **as**13, **as**19 and **as**20 from the group of 12 non-expectation oriented postulates were directly used in forming 9 non-expectation oriented statements.

Therefore, amongst all of the five analyzed theories of rational economic expectations, Sargent's theory of rational economic expectations has the greatest degree of usage of its assumptions in formulating statements resulting from the theory, as well as the highest share of expectation oriented assumptions in the number of all assumptions.

Bibliography

Anderson, P. A. (1979). Rational expectations forecasts from nonrational models. *JME*, pp. 67–80.

Barro, R. J. (1976). Rational expectations and the role of monetary policy. *JME*, pp. 1–33.

Begg, D. K. H. (1982). *The rational expectations revolution in macroeconomics.* Phillip Allan, Oxford.

Brems, H. (1968). *Quantitative economic theory.* Wiley, N.Y.

Cagan, P. (1956). *The monetary dynamics of hyper-inflation.* M. Friedman (ed.), Studies in the quantity theory of money, UCHP, Chicago.

Fischer, S. (1980). *Rational expectations and economic policy.* UCHP, Chicago.

Holden, K. et al., (1985). *Expectations: theory and evidence.* Macmillan, London.

Keynes, J. M. (1936). *The general theory of employment, interest, and money.* Macmillan, N.Y.

Lucas, R. (1972). Expectations and the neutrality of money. *JET*, Vol. 4, pp. 103-124.

Lucas, R. E. (1973). Some international evidence on output-inflation tradeoffs. *AER*, Vol. 63, No 3, pp. 326–334.

Lyndon, R. C. (1966). *Notes on logic.* D. van Nostrand, N.Y.

Minford, A. P. L., Peel, D. A. (1983). *Rational expectations and the new macroeconomics.* Martin Robertson, Oxford.

Modigliani, F. et al. (1958). Forecasting uses of anticipatory data on investment and sales. *QJE*, pp. 23-54.

Muth, J. (1961). Rational expectations and the theory of price movements. *Econometrica*, pp. 315–335.

Nerlove, M. (1958). Adaptive expectations and cobweb phenomena. *QJE*, Vol. 73, pp. 227–240.

Sargent, T. J. (1973). Rational expectations, the real rate of interest, and the "natural" rate of unemployment. *Brookings Papers No 2*, Brookings Inst.

Sargent, T. J., Wallace, N. (1975). "Rational" expectations, the optimal monetary instrument, and the optimal money supply rule. *JPE*, Vol. 83, No 2, pp. 241–254.

Sargent, T. J. (1993). *Bounded rationality in macroeconomics.* Clarendon Press, OUP, Oxford.

Simon, H. A. (1958). *The role of expectations in an adaptive or behavioristic model.* Bowman M. (ed.), Expectations, Uncertainty, and Business Behavior, SSRC, N.Y.

Suppe, F. (1972). Theories, their formulations and the operational imperative. *Synthese*, T. 25, pp. 129-164.

Taussing, F. (1921). Is market price determinate? *QJE*, pp. 396–397.

Varian, H. R. (1984). *Microeconomic analysis.* W. W. Norton, N.Y.

Wójcicki, R. (1974). *Metodologia formalna nauk empirycznych.* Ossolineum, Wroclaw.

Working, H. (1934). A random–difference series for use in the analysis of time series. *JASA*, pp. 11–24.

Working, H. (1949). The investigation of economic expectations. *AER*, Vol. 39, No 3, pp. 150–166.

Working, H. (1958). A theory of anticipatory prices. *AER*, Vol. 48, No 2, pp. 188–199.

Chapter 8

Methodological problems of choosing the instruments and effects of monetary policy

Summary

This chapter contains an exploration of possible means and ends of monetary policies as conducted by central banks. The literature on this subject is, even when restricted just to the English language, very rich and varied with respect to the scope of studies, their spectrum from abstractness to empiricism, precision of reasoning, as well as in the theoretical or practical usefulness of the conclusions. Here, we only discuss certain tools and effects of monetary policies. These effects are strictly connected with the aims of these policies, which are discussed in the introduction to this chapter. Six types of monetary policy tools and twelve groups of effects of such policies are considered. These groups of tools and effects are not exhaustive, both in terms of studying past socio-economic events, as well as the eventual study of future events. Most of included reflections and remarks were originated from both the literature enclosed with this chapter and author own empirical experience of the last 25 years in meso- and macromodeling of Polish economy.

Keywords: monetary policy, rules of monetary policy, discretion in monetary policy, instruments of monetary policy, effects of monetary policy.

8.1. Introduction

In the XX c., XXI c. both consumption and financial goods price inflation has been a central focus of interest to economists, politicians, businessmen, consumers and investors. The peak inflation levels, during the financial crisis in the first half of XX c., have enforced the USA, UK and later other countries governments and parliaments to introduce the law acts that radically increased the role of CB (i.e. Central Banks) and their monetary policy rules in influencing the states of considered economies. One of the results of these acts was the increase of the numbers of research projects that were aimed at studying effects of using different policy instruments (by the FRB of USA, and later main CBs of Europe, Canada, Australia) to impact inflation levels. It turned out that misuse of the policy tools can be harming for employment of different forms of capital, as well as, for real output, incomes of different groups of participants in the considered economies. This is why we treat problems of empirical choice of the statistical measures of policy instruments and their effects as the fundamental part of successful macromodeling of studied economies, as well as, forecasting their future growth.

The concept of monetary policy covers the set of actions of the Central Bank (CB) (for a given country and period of time). These actions are performed in order to realize chosen goal(s) of the CB. The final actions of the CB take the form of formal decisions of the appropriate statutory organs of the CB, such as the CB-Board, CB-council or CB-chairmen, acting in the name of the CB, after establishing a joint position on policy formulation. These decisions are preceded by the evaluation of appropriate statistical data, preparatory research studies, discussions between the members of the appointed organs, the research staff of the CB and/or other researchers.

It should be stressed that, in any country, the selection of candidates for the most influential posts in the CB is carried out by the most influential persons from the ruling party (parties), financial institutions and intelligence agencies. The margin for independence in decisions on CB-monetary policy varies from country to country, and is determined by the actual configuration of the balance of power in the institutions mentioned above and their leaders, as well as their relative power to influence each other. Besides these factors, there are other factors comprising internal and foreign financial institutions (such as banks, stock exchanges, social security funds, insurance companies), trade and industry companies, media agencies (TV, internet, newspapers), and the activities of foreign governments. It is common knowledge that the objectives or roles of the CB are as follows:

o1) to be a depository for reserves of money, and for clearing the balances of depository institutions,

o2) to be a provider of additional currency, as well as credit for banks and other depositories,

o3) to coordinate necessary reforms of the financial system,

o4) to superintend the functioning of the financial system,

o5) to organize and perform open market operations (OMO) when necessary to fulfill the requirements of monetary policy,

o6) to be a depository, fiscal agent and safety-tap for the government treasury, as well as performing important subtreasury operations,

o7) to maintain, develop and stabilize the domestic money market,

o8) to be (if needed) a participant in coordinating actions of the International Monetary Fund (IMF) and World Bank (WB) aimed at stabilizing world financial markets,

o9) to keep price inflation either close to a target level or in a target interval,

o10) to sustain economic growth close to a target level or in a target interval.

While aims o1–o7 belong to the group of aims that CB fulfills within the usual strict role of realizing monetary policies, aims o9–o10 go beyond these roles. The realization of goal o8 is sometimes not in the interest of a given economy or its banking system. If fully realized, the last three objectives may cause many problems both in realizing the monetary policy of the CB and the fiscal policy of the government, due to the inducement of endogenous and exogenous factors shaping the behavior of consumers, investors, households and politicians. Reactions may be especially strong if CBs, such as the Federal Reserve Bank (FRB), European Central Bank (ECB), Central Bank of China (CBCh), Bank of Japan (BJ), Bank of England, have to radically change their policies concerning o8–o10.

For many reasons, studies of the degrees to which particular objectives in the set o1–o10 are achieved, both individually and as a group, are very difficult. This is due, among other reasons, to a lack of reliable, complete data for the objects studied that cover long periods and a large number of economic variables. For example, it is hard to find and gather data concerning profits, flows of different forms of financial capital, the degree of utilization of different forms of capital, productivity of capital, or the expectations of consumers, investors, households and politicians. This will be clearer after a presentation of both the tools and the effects of monetary policy.

The scope of the text is limited to the possible effects of decisions made by the CB connected with aims o1, o2, o5, 06, 07, o9 and o10.

Section 2 describes six sets of instruments of monetary policy, denoted t1–t6, where t2 and t6 are direct tools, and t1, t3, t4 and t5 are indirect tools.

Section 3 contains a presentation of the possible effects of monetary policy as seen in many economic, financial, and social variables. The list of these variables includes 10 groups.

It should be underlined that the literature on problems related to monetary policy is very rich in all major languages, especially in English. This literature includes monographic papers, books, reports of CBs, or parts of books or articles devoted to more general problems of economic analysis. The following books and papers remain inspirational and instructive in the development of modern theories and applications of monetary theory: W. Jevons (1875), E. Böhm-Bawerk (1891, 1906), J. B. Clark (1888), K. Menger (1892), I. Fisher (1930), K. Wicksell (1936), J. Keynes (1936), J. Hicks (1935) and D. Robertson (1922). However, these authors also used older ideas and empirical facts. Some of these ideas had already been expressed in classical Greece, e.g. by a famous poet (see "Frogs" by Aristophanes who underlines "abuse in the choice ... of coins for common use made of ... useless copper" [instead of] the old truly valuable gold coins), or in Xenophon's work "Oikonomikos", and, in particular, by Aristotle in his books on ethics, politics and economics. In the case of Aristotle, we may find a description of the roles of money such as: a unit of measurement of value for goods to be exchanged, a medium of exchange, a store of value and legal source of purchasing power, a legal nomisma (coinage). Notes on the relation between money accumulation and the types of socio – political systems present in antiquity and types of state income, expenditure, and the relative role of trade and financial speculation in the practice of market makers are also particularly interesting in this regard. To an interested reader who wishes to study a broader discussion of authors from the Middle Ages and Renaissance, such as Oresme and Gresham, the books of T. W. Balch (1908) and F. W. Fetter (1932) are highly recommended. In this respect, it is important to recall the famous writings (in Latin and Polish) of M. Kopernik (N. Copernicus):

N. Copernicus (1517): "N. C. Meditata",

N. Copernicus (1519): "Tractatus de monetis",

N. Copernicus (1519): “De estimatione monete” (also edited in German as “Modus cudendi monetam”),
N. Copernicus (1528): “Monete cudente ratio”.
N. Copernicus writings formulated the following views, among others:

φ1) the degradation (vilitas – cheapness, low price, demeaning, debasement) of a currency leads to extinction (misfortune, pestilence, loss, perdition) for a country,

φ2) φ1 acts slowly, secretly and non-violently,

φ3) coinage is a universal measure of the value (of goods),

φ4) the intrinsic value of a coin, $\widehat{val}\,(coin)$, is different to its nominal (face) value, $val\ (coin)$,

φ5) $\widehat{val}\,(coin) \downarrow$ if the number N_c of coins grows too fast,

φ6) $\widehat{val}\,(coin) \downarrow$ if the share of gold or silver in it is lowered,

φ7) It is erroneous to introduce into circulation new “bad” coins when there are simultaneously “good” coins of the same nominal value, but higher intrinsic value, still in circulation, because the worse coins would infect the good coins, and even more, the worse coins would drive out the good coins, i.e. they would win the competition with (destroy, force out, oust, supplant, expulse) good coins,

φ8) In Prussia, the process of bad coins ousting good coins was constant, i.e. the process of Moneta “expugnavitium” was long-term and occurred without any interruptions; the main questions posed were: “what would happen with coinage in the future? How should the circulation of money be reformed?

φ9) Bad coins do not help poor or medium-income people, but they promote laziness in society; it is better to have uniform coinage provided by one mint,

φ10) It is good to have a small number of different coins, each with their own nominal value.

Note 8.1. The law that coins made from valuable metal have a tendency to be replaced by cheaply made coins is now known as the Copernicus-Gresham law. Thomas Gresham lived in 1519–1579, so he could not have written on this subject before 1530. However, as Edward VI's financier in the 1550s, he used such principles to great practical effect in wiping out the king's debts at the expense of his creditors.

A question arises as to whether Copernicus' views expressed in $\varphi1 - \varphi10$ remain correct, not only in the case of money, but also for other goods, e.g. consumption and investment goods, or even for statistical data, scientific models and theories.

8.2. Tools of monetary policy

Money is a specific commodity (token, instrument) that can be used in the following roles (see, e.g. W. Jevons (1875), K. Wicksell (1936), E. Böhm-Bawerk (1891) and I. Fisher (1930), where some of the roles are discussed):

mr1) as a medium for the exchange of goods and services,

mr2) as a store of value and source of purchasing power,

mr3) as a unit of economic and financial accounts,

mr4) as a medium and units of payments (current or deferred),

mr5) as a form and medium of market liquidity,

mr6) as a unit of measurement of various kinds of capital (physical, financial, human, social, cultural, intellectual) and natural resources (including human-beings),

mr7) as a legal tender, nomisma, introduced by the state.

There have been many forms of money in the history of mankind. In ancient Egypt, China, Greece, Italy, Israel, Assyria, Persia and India, money took the form of metallic coins (silver, gold, bronze, copper, or metal alloys). Metal alloy coins remain in use today. In the XVIII and XIX c., paper money was used simultaneously with metallic money in France, USA, UK, Italy and Germany. Later, in the XX–XXI c., paper money expulsed metallic coins, in accordance with N. Copernicus' law. It is possible that electronic-accountance money will oust paper money. In the XX c., paper money took the form of government notes or banknotes. The notes issued by a government's mints are secured by either gold reserves or governmental security papers or take the form of unsecured fiat money. Banknotes, in turn, are secured by gold reserves, government securities, commercial papers, general assets or combinations of these assets.

From the point of view of durability and malleability, metallic coins are superior. However, the portability, divisibility, recognizability and homogeneity of paper money favor it in comparison to metallic coins.

The roles mr1–mr7 vary over time and space, as well as with respect to technical progress (the introduction of telegraphs, telephones, computers, the Internet, smartphones, facebook, twitter, television) and the creation of big data sets which are easily accessible to almost every consumer, investor, household, politician, pupil, educator and scientist. Each of these inventions had different implications regarding the behavior of people. Probably the latest advances in information and communication technology have had the greatest force of influence through the hands of investment banks, commercial banks, insurance companies, governmental agencies, international corporations, political parties' committees and journalists selling their services by the use of press, TV portals, activists in new movements and scientists, as well as through the actions of criminals, together with the participants and organizers of black markets. The scope of these impacts depends on the income distribution in a nation, which is partially determined by the structure of society in terms of the shares of people who are, appropriately, homo astrictus (responsible), homo astutus (shrewd), homo avidus (greedy), homo canis (followers), homo faber (artisans), homo ignavus (lazy), homo improbus (dishonest), homo opportunus (opportunistic), homo stultis (lacking in intelligence).

There are other human characteristics that are recognized by psychologists and have influence on homo economicus and changes in the degree of importance of mr1–mr7. These are e.g. the short memory of most people about facts, events and processes, especially about social, political, economic and cultural events, as well as the demographic structure of slow and fast thinking people. Some of these features enable grand scale manipulation of human consciousness and behavior by professional influence animators. It seems that the roles mr2, mr4, mr5 may be especially sensitive to such influences.

The central bank (CB) may have a very important impact on all of the seven roles of money mentioned above. The CB has the following tools of monetary policy:

t1) CB – interest rates,

t2) obligatory reserves of paper money and coins in the CB and other banks, together with CB-reserves in the form of gold, t3) the open market operations (OMO) of CB,

t4) CB – deposit operations,

t5) CB – credit (secured by credit bonds, stocks, certificates of deposit and other securities),

t6) money supply.

The group of t1)-instruments contains:

t11) the CB – lombard interest rate $R^{\lambda}_{cb}(t)$,

t12) the CB – discount and rediscount interest rates $R^{\delta}_{cb}(t)$, $R^{\rho\delta}_{cb}(t)$,

t13) the CB – OMO – interest rate (CB-reference rate), $R^{omo}_{cb,l}(t)$,

t14) the CB – deposit interest rate, $R^{d}_{cb}(t)$.

The CB – lombard interest rate, $R^{\lambda}_{cb}(t)$, is a periodic, percentage, monetary payment for the short-term use of lombard refinance credit loaned by the CB to other banks based on the pledge of specified security papers (such as: bonds, CD, CP, TB and BOE, where CD denotes – certificates of deposits, CP – commercial papers, TB – treasury bills, BOE – bills of exchange). Usually, the value of lombard credit comprises $\frac{3}{4}$ of the value of the pledged paper and the rate $R^{\lambda}_{cb}(t)$ can be very high during periods of mega-inflation. It is always the highest CB – interest rate.

The CB – $R^{\delta}_{cb}(t)$ rate is the R_{cb} interest rate at which banks may borrow monetary funds from CB – reserve money for short periods. This rate is treated as the cost (or price) of borrowing the CB's reserve – money. The CB – discount rate "window" is linked to the CB – rediscount rate "window", and the CB – OMO interest rate, $R^{omo}_{cb}(t)$, is now the most commonly used rate by many CB-s.

The CB – rediscount interest rate, $R^{\rho\delta}_{cb}(t)$, is the $R^{\delta}_{cb}(t)$ discount interest rate that is charged by CB for rediscounting (i.e. discounting for a second time) CDs, CPs, BOEs, industrial or agricultural security papers or other legal securities. Such rediscounting enables banks to obtain refinance rediscount credit from the CB.

The OMO – interest rate, $R^{omo}_{cb,i}(t)$ or $R^{omo}_{l,i}(t)$, are the CB's or bank l's price for purchasing R^{p}_{i} or price for selling R^{s}_{i} of security paper "i" at time t on the open market for money or securities. The subindex "i" concerns e.g. treasury bills, bankers' acceptances, foreign currencies, CB – bills, CB –bonds. Purchases and sales are executed under the conditions of repurchase agreements (so called REPO). A REPO transaction assumes that one party of an agreement sells security "i" to another party, and at the same time, it commits itself to repurchase an identical type of security on a date and at a price specified in the agreement. Thus the borrower delivers securities of type "i" and receives cash from the lender at the repo rate, which remains constant during the period of agreement.

It would be instructive, therefore, to also use the symbols $R_{cb,i}^{p,omo}(t)$ and $R_{cb,i}^{s,omo}(t)$. It can easily be seen that the CB – as the biggest market participant – can strongly influence both of the rates $R_i^{p,omo}$ and $R_i^{s,omo}$. For the theoretical and practical aims of economic and financial analyses, it is useful not only to use the interest rates defined above, but also averages based on them. These include arithmetic averages obtained by averaging over types $\{j\}$ of transactions, types of securities or other financial goods $\{i\}$, banks $\{l\}$, and moments of time $\{t\}$. This means that we can calculate such averages as (the line over the letter R denotes average):

$$\overline{R_l} = \frac{1}{\nu}\sum_{t=1}^{\nu} R_l(t), \overline{R}_{cb}^{\lambda} = \frac{1}{\nu}\sum_{t=1}^{\nu} R_{cb}^{\lambda}(t), \overline{R}_{cb}^{\delta} = \frac{1}{\nu}\sum_{t=1}^{\nu} R_{cb}^{\delta}(t), \overline{R}_{cb}^{\rho\delta} = \frac{1}{\nu}\sum_{t=1}^{\nu} R_{cb}^{\rho\delta}(t),$$

$$\overline{R}_i^{p} = \nu^{-1}\sum_{t=1}^{\nu} R_i^{p}(t), \overline{R}_i^{s} = \nu^{-1}\sum_{t=1}^{\nu} R_i^{s}(t), \overline{R}_i^{omo}(t) = N_{tr}^{-1}\sum_{j=1}^{N_{tr}} R_{i,j}(t),$$

$$\overline{R}_i^{omo} = \nu^{-1}\sum_{t=1}^{\nu} R_i^{omo}(t), \overline{R}^{omo} = m^{-1}\sum_{i=1}^{m} \overline{R}_i^{omo}, \overline{R}_{cb,\ i}^{omo} = \nu^{-1}\sum_{t=1}^{\nu} R_{cb,i}(t),$$

$$\overline{R}_{cb}^{omo} = m^{-1}\sum_{l=1}^{m} \overline{R}_{cb,l}^{omo}(t), \overline{R}_{cb,i}^{omo} = N_{tr}^{-1}\sum_{j=1}^{N_{tr}} \overline{R}_{cb,i,j}^{omo}.$$

Besides these CB – rates, there is also the CB – deposit interest rate $R_{cb}^{d}(t)$, i.e. the CB – interest rate that is paid by the CB on various types of deposit accounts held by other banks, e.g. overnight, week, 2 week BC – rates of interest. Recently, the rate $R_{cb}^{d}(t)$ has most often determined the lower bound (limit) on market fluctuations in the overnight rate. It should be stressed that the rates $(R_{cb}^{\lambda}(t),\ R_{cb}^{d}(t))$ are used to define the corridor of fluctuations in the overnight market rate of interest which is symmetric with respect to $(\overline{R}_{cb}^{omo}(t),\ \overline{R}_{cb,i}(t))$.

Collecting the possible interest rate tools of the CB together at moment t, we obtain the following configuration (vector) of CB – interest rates:

$$\mathbb{R}_{cb}(t) = \left(R_{cb}^{\lambda}(t), R_{cb}^{\delta}(t), R_{cb}^{\rho\delta}(t), R_{cb}^{d}(t), R_{cb,\{i\}}^{p,omo}(t), R_{cb,\{i\}}^{s,omo}(t)\right). \quad (8.1)$$

Note 8.2. In empirically based models for the optimal control of monetary policy, many authors use either $R_{ct}^{\rho\delta}$ or R^{LIBOR} on the assumption that they accurately reflect both the direction and scale of changes in the monetary policy of the FRB or Bank of England. The author is not aware of any publications using 8.1 as a basis for formulating monetary policy variables (treated here as steering variables).

Note 8.3. In studying the effects of the monetary policy of the CB, it would be important to empirically establish the degrees of impact of particular component interest rates from 8.1, as well as the whole collection of rates from 8.1, not only, e.g. on the market rates LIBOR, LIBID, FIBOR, FIBID, WIBOR and WIBID, but also on the volumes of transactions on stock exchanges and foreign exchanges (FOREX), as well as other economic quantities in both financial and non-financial spheres.

The second kind of instrument, t2), of CB – monetary policy is the legally required value of reserve money in the form of compulsory domestic and foreign currency deposits by depository financial institutions in CB – accounts, as well as in the form of a percentage of the monetary value of the turnover from transactions involving securities (including OMO – repo, sell-buy-back transactions between the CB and other banks, saving – credit cooperative banks). In practice, representatives of CB-s, economists and politicians all prefer using the concept of a minimal reserve ratio, i.e. the ratio of the monetary value of reserves to deposits and trade volume for securities. It should be underlined that different CB-s use different practical definitions of the ratio further denoted here by $MRR_{cb}(t)$. These measures vary in their universality, uniformity, frequency of use, effectiveness in shaping and maintaining the liquidity of the financial sector, degree of differentiation between domestic and foreign deposits, assets or liabilities.

Note 8.4. Changing $MRR_{cb}(t)$ drastically has significant deleterious effects on the performance of the financial sector, and through this, on the whole of an economy. Milton Friedman once proposed to fix $MRR_{cb}(t) = 100\%$. However, reasons of prudentiality suggest that during booms or active phases of the business cycle this ratio should even be as low as 30%.

The third family, t3), of monetary tools embraces both the interest rates already discussed, $\left\{R_{cb,\, i}^{omo}(t)\,,\ \overline{R}_{cb}^{omo}(t)\,,\ \overline{R}_{cb}^{omo}\right\}$, and CB purchases and sales of government securities (treasury bills and bonds, as well as other short-term securities), bankers' acceptances, foreign currency transactions, e.g. cable transfers, swaps that are executed in the form of repurchase agreements. The OMO helps the CB to contract or expand the volume of excess reserves within the banking system and affect its credit rating, to help the government in its debt trading, to affect FOREX markets (if such operations are carried out by the FRB, ECB, BE, CBCh or BJ) and to ease the mobility of financial capital, K_f. OMO – operations, together with the rates $\{R_{cb}^{omo}\}$, are considered by many economists to be the most flexible instruments of CB – monetary policy, although they do not have such uniform and strong effects, according to them, as the appropriate changes in legal reserve requirements imposed on banks.

The fourth group t4) of CB – monetary tools involve the interest rate for deposits made by other banks, the government and possibly other financial institutions. By using the CB – deposit interest rate, R_{cb}^{d}, included in the system 8.1 of interest rates, the CB can additionally induce changes in the liquidity of particular financial institutions, causing it to fall or increase, as appropriate.

The fifth t5) group of CB tools consists of the CB giving loans (credit) to the government, banks or other financial institutions. Granting such credit depends upon the CB's confidence in a particular debtor based on an evaluation of the character of their business, attitude to risk, capacity to repay (business risk) and collateral capital (property risk). The FRB, for example, provides loans in the form of short-term adjustment credit and extended credit to depository financial institutions, in order to enable borrowers to meet short-term business needs (not needs for speculation). The FRB and its borrowers base their operations on the spread between R_{cb}^{δ} and market loan rates R^c, where "c" denotes credit. Extended credit is given to banks that serve important goals of public interest, but have problems with obtaining funds for their activities. Here, R_{cb}^{δ} denotes the cost of credit usually without surcharges and with more lenient repayment schedules.

The sixth t6) group of CB – tools covers money supply aggregates. By money supply, we understand the amount of money (cash) in circulation (coinage and paper money), together with bank deposits (i.e. bank reserves that are readily accessible to customers, although this is not treated as money but as credit papers). There are no universally accepted classifications of monetary aggregates, even for a single economy, groups of economies or relatively short period of time. The most often used labels for monetary aggregates are M0, M1, M2 and M3.

In Great Britain, M0 is given by the value of notes and coins in circulation plus banks' non-statutory deposits with the Bank of England. In Germany in the 1990s, M0 covered the cash in circulation and the reserve money held by banks. In the USA, M0 denoted the liabilities of the FRB and its currency, including the deposits of member banks, which together were called the monetary base. It is often accepted that by performing open market operations or changing the required reserve ratio, a CB changes the money supply in an economy.

Recently, the aggregates M1, M2 and M3 have been more popular in defining monetary policies. The ECB's definitions of the monetary aggregates M1-M3 in the Euro zone are given[1] in Table 1.

Table 8.1. Liabilities of the money – issuing sector and central government of a monetary character and held by the money – holding sector

	M1	M2	M3
Currency in circulation	X	X	X
Overnight deposits	X	X	X
Deposits with an agreed maturity period of up to 2 years		X	X
Deposits redeemable with a period of notice up to 3 months		X	X
Repurchase agreements			X
Money market fund (MMF) shares/units			X
Debt securities up to 2 years			X

Source: European Central Bank

Note 8.5. M1, as defined in Table 1, is called "narrow money". The aggregate M2 is referred to as "intermediate money". The deposits within M2 are treated as liquid. The aggregate M3 is called "broad money". The last three components of M3, due to their high degree of liquidity and price stability, make these component instruments of M3 close substitutes of deposits and stabilizers for the aggregate M3. These definitions are at present generally respected in EU countries that do no belong to the Euro zone.

In the USA, at present M1 covers currency and demand deposits (since 1971), M2 includes M1 plus fixed time deposits, saving deposits and non-institutional MMFs (Money Market Funds). Thus, M2 is equivalent to the previous definition of M3, and the current definition of M3, given by M2 + deposits in mutual savings banks, savings and loans with credit unions, resembles the previous definition of M5. This shows that changes in the definitions of particular aggregates make it difficult, not only to trace the historical evolution of these aggregates, but also even to make economic and

[1] See: https://www.ecb.europa.eu/stats/money/aggregates/aggr/html/hist.en.html.

financial comparisons of the effects of monetary policies in a single country using cross – sectional time – series data covering a long period of time.

To see how uncertain international comparisons would be in this respect, it suffices to compare how different the definitions of these aggregates are and how often they are changed in such countries as Australia, Canada, Japan, South Korea, India, China, Brazil and Russia.

It should be remembered that orthodox monetarists criticized the use of multiple targets, due to the non–transparency of their effects. It was P. Volcker who successfully introduced a procedure to use non–borrowed reserves to control the growth of M1, M2 in order to reduce inflation. After this, the FRB's monetary policy did not give a central role to monetary or credit aggregates, despite Milton's recommendation to link growth in the money supply with output growth being generally accepted. However, a fashionable policy trend of deregulation, financial innovation, the growing influence of the Internet and new telecommunications devices has given open access to data, together with increased manipulation of society's expectations and of the behavior of investors on financial markets, especially on the influential New York and London Stock Exchanges and National Association of Securities Dealers Automated Quotations (NASDAQ). In the mid 1970s, when the Federal Open Market Committee (FOMC) was using guidelines for growth in monetary supply, estimates of M1 based on demand equations were not useful, due to the resulting inaccurate forecasts of monetary growth, that resulted, as stated by S. Goldfeld, from "the case of the missing money". Probably, mainly financial innovations had destabilized the economic relationships between the discussed monetary aggregates and other economic quantities that are the effects of the use of tools of both monetary and fiscal policy. These symptoms of instability can also be observed in the year to year percentage changes in the M1 supply. The biggest fluctuations occurred in Japan (in the periods between successive recessions) and the second biggest in the UK (from 1988). The USA takes third place in this ranking (starting in 1988) and the fluctuations in M1 growth were smallest in the Euro zone, which started to increase after 2004. The biggest increases in M1 volumes are observed (after 2002) in Japan, and then, in decreasing order, for the Euro zone, USA, UK. The results for M2 growth after 2008 are similar.

It would be worthwhile to carry-out broad studies of the hypothesis that large changes in M1, M2 and M3 in the Japan, the UK, the Euro zone countries, as well as China, India, Brazil, South Africa, Russia, Canada, Australia and Poland were affected by the structure of their balance of payments, their exchange rates with the US dollar, dictated by the fact that between 45% and 65% of US currency is circulating abroad, and, in particular, the growing number of financial instruments. For further information see, among others, F. Bredon and P. Fisher (1996), T. Sargent, et al. (1973), E. Phelps, J. Taylor (1977), K. Wicksell (1907), R. Barro (1976), S. Fischer (1975), R. Fair (1994), W. Poole (1970), C. Woelfel (1994), M. Choudhry (2001), J. Tobin (1998), A. Elbourne, J. de Haan (2009), C.D. Romer, D.H. Romer (1997), B.M. Friedman (ed.) (1999), M. Brzoza-Brzezina (2011), M. Górajski (2017), M. Górajski, M. Ulrichs (2016), W. Milo et al. (2002, 2003, 2006, 2010) and the literature included in these works.

8.3. Effects of monetary policy tools

It is important, when studying the roles of monetary policies, to distinguish between the following kinds of effects of monetary policy tools:

e1) observable and non – observable,

e2) measurable and non – measurable,

e3) quantitative and qualitative,

e4) equilibrium and disequilibrium inducing,

e5) stabilizing and destabilizing: with regard to production, exchange, distribution, circulation, accumulation of goods,

e6) evoking and not evoking: financial, economic and social turbulence,

e7) synergetic and non – synergetic with the effects of taxation policy, i.e. inducing stronger and weaker joint effects on the state of the economy,

e8) synchronous and asynchronous with the effects of the taxation and custom policies of a government,

e9) pro – growth and anti – growth: with respect to the economy,

e10) short – term, medium – term and long – term,

e11) micro – level, meso – level and macro – level in their influence,

e12) permanent and transitory,

e13) stimulating and restricting: innovation, entrepreneurship, competition in the domestic economy, e14) strengthening and weakening: economic robustness to shocks in demand and supply.

The above list of types of effects may be extended. Some of the descriptive characteristics of effects e1–e14 can be ascribed to the economy studied (or part of it) in a given period relatively easy. This applies to the characteristics of effects e1–e3. Practical use of the types of effects e4–e12 needs further explanation and argument as regards non – uniqueness of definition, as well as the quality of measurement for both the tools used and the chosen forms of effects of monetary policies.

Note 8.6. There are general and particular dependencies between pairs, triples and larger subsets of $\{ei\}$, $i = 1, \ldots, 14$. These dependencies have both theoretical and empirical forms. It should be noted that due to constraints on statistical data, many theoretical and empirical pronouncements are more imaginative, hypostatic and/or hypothetic in form than judgements which are empirically well confirmed on the basis of clear and extensive data sets. ■

The 14 types of effects of monetary policy presented above can be linked to or observed in many factual economic, financial, social, political, legal and cultural events or processes that take quantitative or qualitative forms. Our attention, henceforth, will focus on economic and financial effects. Thus we assume that a CB may use any of its monetary tools $\{tj\}$, $j = 1, \ldots, 6$, to realize the following type of aims:

a1) controlling nominal price inflation $\pi(t) \equiv \dot{P}(t) \equiv \frac{\triangle P(t)}{P(t-1)}$, so that it lies within a given interval:

$$\begin{aligned}
\text{I.}\ & \dot{P}(t_a+\tau) \in [\dot{P}_{cb}(t_a+\tau),\ (1+\alpha)\dot{P}_{cb}(t_a+\tau)],\\
\text{II.}\ & \dot{P}(t_a+\tau) \in [\dot{P}_{cb}(t_a+\tau),\ (1+\alpha)\dot{P}^{ex}_{cb}(t_a+\tau)],\\
\text{III.}\ & \dot{P}(t_a+\tau) \in [\dot{P}_{cb}(t_a+\tau),\ (1+\alpha)\ \dot{P}^{ex}(t_a+\tau)],\\
\text{IV.}\ & \dot{P}(t_a+\tau) \in [\dot{P}_{cb}(t_a+\tau) - \varepsilon_{\text{cb}}, \dot{P}_{cb}(t_a+\tau) + \varepsilon_{\text{cb}}],\\
\text{V.}\ & \dot{P}_{cb}(t_a+\tau) \in [\dot{P}^{ex}_{cb}(t_a+\tau) - \varepsilon_{\text{cb}}, \dot{P}^{ex}_{cb}(t_a+\tau) + \varepsilon_{\text{cb}}],\\
\text{VI.}\ & \dot{P}_{cb}(t_a+\tau) \in [\dot{P}^{ex}_{cb}(t_a+\tau),\ (1+\alpha)\dot{P}^{ex}_{cb}(t_a+\tau)],\\
\text{VII.}\ & \dot{P}_{cb}(t_a+\tau) \in [\alpha\dot{P}_{cb}(t_a+\tau),\ \ (1+\alpha)\dot{P}_{cb}(t_a+\tau)],
\end{aligned}$$

where $\triangle P(t) = P(t) - P(t-1)$, t_a denotes the present time, $\tau > 0$, $\dot{P}_{cb} \equiv \pi_{cb}$ – target inflation level fixed by the CB, $\alpha \in [0,2; 0,4]$, "ex" denotes expected or forecasted level of inflation, ε_{cb} fixed change of CB-target inflation, a2) controlling the economic growth rate $\dot{Y}(t) = \frac{\triangle Y(t)}{Y(t-1)}$, $\triangle Y(t) = Y(t) - Y(t-1)$, so that it lies within a given interval:

$$\begin{aligned}
&\text{I. } \dot{Y}(t_a+\tau) \in [\dot{Y}_{cb}(t_a+\tau),\ (1+a)\dot{Y}_{cb}(t_a+\tau)],\\
&\text{II. } \dot{Y}(t_a+\tau) \in [\dot{Y}_{cb}(t_a+\tau),\ (1+a)\ \dot{Y}^{ex}(t_a+\tau)],\\
&\text{III. } \dot{Y}(t_a+\tau) \in [\dot{Y}_{cb}(t_a+\tau) -\ \eta_{cb},\ \dot{Y}_{cb}(t_a+\tau)+\eta_{cb}],\\
&\text{IV. } \dot{Y}_{cb}(t_a+\tau) \in [\dot{Y}^{ex}_{cb}(t_a+\tau) -\ \eta_{cb},\ \dot{Y}^{ex}_{cb}(t_a+\tau)+\eta_{cb}],\\
&\text{V. } \dot{Y}_{cb}(t_a+\tau) \in [\dot{Y}^{ex}_{cb}(t_a+\tau),\ (1+a)\dot{Y}^{ex}_{cb}(t_a+\tau)],\\
&\text{VI. } \dot{Y}_{cb}(t_a+\tau) \in [a\dot{Y}_{cb}(t_a+\tau),\ (1+a)\dot{Y}_{cb}(t_a+\tau)],
\end{aligned}$$

where the symbols used here have analogous roles to those used above concerning price inflation, but correspond to the production growth rate $\dot{Y}$, e.g. GDP, GNP, or NI (national income);

a3) impacting the range of average values of CB-exchange rate values $\bar{E}_{cb}(t_a+\tau)$, $\tau > 0$, to belong to the optimal range $[E^*_{cb}(t_a+\tau), (1+\alpha E^*_{cb}(t_a+\tau))]$, where optimality is defined with respect to the CB-chosen objective function, f. ex. for chosen α, $\gamma \in [0,1]$,

$$\varphi_1(\gamma, \bar{E}_{cb}(t_a+\tau)) = max\{\gamma BOP(\bar{E}_{cb}(t_a+\tau)) + (1-\gamma)BOT(\bar{E}_{cb}(t_a+\tau))\},$$

$$\varphi_2(\bar{E}_{cb}(t_a+\tau)) = max\{FDI(\bar{E}_{cb}(t_a+\tau))\},$$

and where BOT, BOP, FDI denote balance of trade, balance of payments, foreign direct investments;

a4) restoring and/or sustaining the stability of the financial system of the economy, i.e. constraining the ratio of the total value of credit to the total value of system assets to fluctuate in a defined safety zone;

a5) maximizing the productivity of the total value of the economy's capital K, $K(t) = K_{ph}(t) + K_f(t) + K_h(t) + K_s(t) + K_{iar}(t)$, and, in particular, the productivity of financial capital, where these productivities are given by the ratios $\frac{Y(t)}{K(t)}$, $\frac{Y(t)}{K_i(t)}$, and $i \equiv ph,\ f,\ h,\ s,\ iar$ denotes the following forms of capital goods: physical, financial, human, social, intellectual and authorship rights;

a6) diminishing or augmenting the share of investment or consumption in the aggregate demand of the economy;

a7) diminishing or augmenting the share of profits or wages in the aggregate supply of the economy;

a8) minimizing the degree of non-exploitation of production means;

a9) maximizing various types of productivity, K_i ($i \equiv ph,\ f,\ h,\ s,\ iar$), in the production or service sector of the economy;

a10) aiming simultaneously at ai) and aj); $i, j = 1, \ldots, 9,\ i \neq j$, or ai), aj) and ak), $i, j, k = 1, \ldots, 9,\ i \neq j \neq k$ and so on.

a11) The realization of aims a1)–a10) are, to varying degrees, observable, measurable, quantifiable and equilibrium inducing. Additionally, they stabilize particular parts of the economy and, to some extent, produce effects e6)–e10). Since the estimates of each K_i, $i \equiv ph,\ f,\ h,\ s,\ iar$, are very uncertain, and the estimates of Y are also uncertain, on both the demand and supply side, the estimates of the productivity or generation ability of each type of capital (or total capital) are also uncertain.

There exists a vast literature devoted to a description and discussion of some of the above aims and their manifestations in data sets. Among others, the following are recommended to interested readers: R. Froyen and A. Guender (1998); G. Eggertsson and M. Woodford (2003); M. Woodford (2003), B. Friedman (1998), W. Milo, Z. Kozera, et al. (2003), W. Milo et al. (2006), W. Milo et al. (2010). The first four of these works mainly reference aims a1) - a2), and the practical effects of monetary policy manifested in the empirical facts corresponding to these aims. The last three consider aims a4) and a5), as well as a1) and a2), and empirical facts which either confirm or do not confirm that these aims have been realized.

Traces of the effects of the use of monetary policy tools may be found not only in the above discussed aims and the corresponding quantities, but also in many other quantities which are connected with the objectives a1)–a10) to varying degrees. This implies that, for example, each quantity appearing in the formulations of a1)–a9), can be expressed by a decomposition into two components, a cb – part, which is achieved only due to the action of the monetary policy tools of the CB, and the off – part, which is achieved due to the actions of other forces or factors. For example, in the case of inflation, we have:

$\Pi(t) = \widehat{\Pi}_{cb}(t) + \widehat{\Pi}_{off}(t)$, where "ˆ" denotes that the corresponding quantity is estimated by the use of an appropriate model. For $t < t_a$ this estimate would be an ex post forecast, and for $t > t_a$ it would be an ex ante forecast, since t_a denotes the present moment of time, where $\widehat{\Pi}_{cb}$ denotes the CB – generated part of Π, and $\widehat{\Pi}_{off}$ denotes the part of Π generated by other forces and factors.

Using decompositions of other economic quantities that are affected by tools of monetary policy, we can derive the following estimates: $\hat{Y}_{cb}(t)$, $\widehat{\dot{Y}}_{cb}(t), \hat{E}^{w}_{cb}(t)$, $\hat{U}^{K}_{cb}(t)$, $\hat{U}^{K_i}_{cb}(t)$, $i \equiv ph,\ f,\ h,\ s,\ iar$. In more detail, we may define the following effects of the use of CB – monetary policy tools:

ea1) $\widehat{\Pi}_{cb}(t)$, $\widehat{\dot{\Pi}}_{cb}(t), \widehat{\Pi}^{ex}_{cb}(t), \widehat{\Pi}^{eq}_{cb}(t), \widehat{\Pi}^{pot}_{cb}(t), \widehat{\Pi}^{end}_{cb}(t), \widehat{\Pi}^{st}_{cb}(t), \widehat{\Pi}^{rob}_{cb}(t)$, where: ex≡expected, eq≡equilibrium, pot≡potential, end≡endurable, st≡stable, rob≡robust;

ea1i) $\widehat{\dot{\Pi}}^{+}_{cb}(t), \widehat{\dot{\Pi}}^{-}_{cb}(t), \widehat{\Pi}^{zb}_{cb}(t), \widehat{\Pi}^{cpi}_{cb}(t), \widehat{\Pi}^{ppi}_{cb}(t), \widehat{\Pi}^{gdpd}_{cb}(t), \widehat{\Pi}^{fgpi}_{cb}(t)$, $\widehat{\Pi}^{opt}_{cb}(t)$, where: $+\equiv$accelerating, $-\equiv$ decelerating, zb≡zero bounded, cpi≡consumer price index, ppi≡producer price index, gdpd≡GDP-deflator, fgpi≡financial goods price index, opt≡optimal;

ea2) $\hat{Y}_{cb}(t), \hat{Y}^{ex}_{cb}(t), \hat{Y}^{eq}_{cb}(t), \hat{Y}^{pot}_{cb}(t), \hat{Y}^{end}_{cb}(t), \hat{Y}^{st}_{cb}(t), \hat{Y}^{rob}_{cb}(t)$;

ea2i) the rates of growth of the quantities from ea2);

ea2ii) $\hat{Y}^{d}_{cb}(t), \hat{Y}^{d,ex}_{cb}(t), \hat{Y}^{d,eq}_{cb}(t), \hat{Y}^{d,pot}_{cb}(t), \hat{Y}^{d,end}_{cb}(t), \hat{Y}^{d,st}_{cb}(t)$, $\hat{Y}^{d,rob}_{cb}(t)$;

ea2iii) $\hat{Y}^{s}_{cb}(t), \hat{Y}^{s,ex}_{cb}(t), \hat{Y}^{s,eq}_{cb}(t), \hat{Y}^{s,pot}_{cb}(t), \hat{Y}^{s,end}_{cb}(t), \hat{Y}^{s,st}_{cb}(t)$, $\hat{Y}^{s,rob}_{cb}(t)$;
I – expenditure on private investment, G – government expenditure, A – amortization+materials+energy, W – wages, Z – profits, d – demand, s – supply, Y^d – aggregate demand, Y^s – aggregate supply;

ea2iv) $\frac{\hat{C}_{cb}(t)}{\hat{Y}^{d}_{cb}(t)}, \frac{\hat{I}_{cb}(t)}{\hat{Y}^{d}_{cb}(t)}, \frac{\hat{G}_{cb}(t)}{\hat{Y}^{d}_{cb}(t)}, \frac{\widehat{BOT}_{cb}(t)}{\hat{Y}^{d}_{cb}(t)}$;

ea2v) $gr\left(\frac{\hat{C}_{cb}(t)}{\hat{Y}^{d}_{cb}(t)}\right), gr\left(\frac{\hat{I}_{cb}(t)}{\hat{Y}^{d}_{cb}(t)}\right), gr\left(\frac{\hat{G}_{cb}(t)}{\hat{Y}^{d}_{cb}(t)}\right), gr\left(\frac{\widehat{BOT}_{cb}(t)}{\hat{Y}^{d}_{cb}(t)}\right)$, where $gr \equiv$ growth rate;

ea2vi) $\frac{\hat{A}_{cb}(t)}{\hat{Y}^{s}_{cb}(t)}, \frac{\hat{W}_{cb}(t)}{\hat{Y}^{s}_{cb}(t)}, \frac{\hat{Z}_{cb}(t)}{\hat{Y}^{s}_{cb}(t)}$;

ea2vii) $gr\left(\frac{\hat{A}_{cb}(t)}{\hat{Y}^{s}_{cb}(t)}\right), gr\left(\frac{\hat{W}_{cb}(t)}{\hat{Y}^{s}_{cb}(t)}\right), gr\left(\frac{\hat{Z}_{cb}(t)}{\hat{Y}^{s}_{cb}(t)}\right)$;

ea3) $\widehat{\dot{Y}}_{cb}(t), \widehat{\dot{Y}}^{+}_{cb}(t), \widehat{\dot{Y}}^{-}_{cb}(t), \widehat{\dot{Y}}^{d}_{cb}(t), \widehat{\dot{Y}}^{s}_{cb}(t), \widehat{\dot{Y}}^{ex}_{cb}(t), \widehat{\dot{Y}}^{eq}_{cb}(t), \widehat{\dot{Y}}^{pot}_{cb}(t)$, $\widehat{\dot{Y}}^{end}_{cb}(t), \widehat{\dot{Y}}^{rob}_{cb}(t)$, $\widehat{\dot{Y}}^{st}_{cb}(t), \widehat{\dot{Y}}^{opt}_{cb}(t)$;

ea3i) $grgr\left(\frac{\hat{C}_{cb}(t)}{\hat{Y}^d_{cb}(t)}\right), grgr\left(\frac{\hat{I}_{cb}(t)}{\hat{Y}^d_{cb}(t)}\right), grgr\left(\frac{\hat{G}_{cb}(t)}{\hat{Y}^d_{cb}(t)}\right), grgr\left(\frac{\widehat{BOT}_{cb}(t)}{\hat{Y}^d_{cb}(t)}\right);$

ea3ii) $\hat{\dot{Y}}^{K_i^+}_{cb}(t)$ – growth of Y generated by the increase in capital K_i^+ due to monetary policy, where $i \equiv ph,\ f,\ h,\ s,\ iar$;

ea3iii) $\hat{\dot{Y}}^{ser}_{cb}(t)$- $gr(Y)$ generated by the increased share of services due to the CB;

ea4) $\hat{E}^{w}_{cb}(t), \hat{E}^{w,eq}_{cb}(t), \hat{E}^{w,ex}_{cb}(t), \hat{E}^{w,pot}_{cb}(t), \hat{E}^{w,st}_{cb}(t), \hat{E}^{w,rob}_{cb}(t), \hat{E}^{w,end}_{cb}(t)$, $\hat{E}^{w,opt}_{cb}(t)$, $\hat{E}^{w,cc}_{cb}(t), \hat{E}^{w,ff}_{cb}(t)$, where cc ≡crisis contagion part, ff ≡financial fragility part;

ea5) $\frac{\hat{D}^H_{cb}(t)}{\widehat{AS}^H_{cb}(t)}, \frac{\hat{D}^{PF}_{cb}(t)}{\widehat{AS}^{PF}_{cb}(t)}, \frac{\hat{D}^{FS}_{cb}(t)}{\widehat{AS}^{FS}_{cb}(t)}, \frac{\hat{D}^{GS}_{cb}(t)}{\widehat{AS}^{GS}_{cb}(t)}$, where $\hat{D}^H_{cb}$ denotes the total debt of households (H) originating from the CB – monetary policy, $\widehat{AS}^H_{cb}$ denotes the total assets of household created by CB – monetary tools, analogously, $PF \equiv$ denotes private firms, FS ≡financial sector, $GS \equiv$ government sector;

ea5i) $\frac{\widehat{RES}^{FS}_{cb}(t)}{\widehat{AS}^{FS}_{cb}(t)}, \frac{\widehat{RES}^{FS}_{cb}(t)}{\widehat{CR}^{FS}_{cb}(t)}, \frac{\hat{D}^{ec}_{cb}(t)}{\hat{Y}_{cb}(t)}, \frac{\hat{I}^{FS}_{cb}(t)}{\hat{D}^{ec}_{cb}(t)}, \frac{\hat{I}^{FS}_{cb}(t)}{\hat{D}^{ec}(t)}$, where RES^{FS} denotes the value of reserves in the FS – financial system, ec – economy;

ea5ii) $\frac{I^{K_i}(t)}{D^{ec}(t)}, \frac{I^{K_s}(t)}{G(t)}, \frac{\hat{I}_{cb}(t)}{\widehat{Sav}_{cb}(t)}$, Sav ≡total savings;

ea6) $\frac{\hat{Y}_{cb}(t)}{\hat{K}_{cb}(t)}, \frac{\hat{Y}_{cb}(t)}{\hat{K}_{cb,i}(t)}, \frac{\hat{Y}^{K_i}_{cb}(t)}{\hat{K}_{cb,i}(t)};$

ea6i) $\frac{\hat{Y}^{ex}_{cb}(t)}{\hat{K}^{ex}_{cb}(t)}, \frac{\hat{Y}^{eq}_{cb}(t)}{\hat{K}^{eq}_{cb}(t)}, \frac{\hat{Y}^{end}_{cb}(t)}{\hat{K}^{end}_{cb}(t)}, \frac{\hat{Y}^{pot}_{cb}(t)}{\hat{K}^{pot}_{cb}(t)}, \frac{\hat{Y}^{st}_{cb}(t)}{\hat{K}^{st}_{cb}(t)}, \frac{\hat{Y}^{rob}_{cb}(t)}{\hat{K}^{rob}_{cb}(t)}, \frac{\hat{Y}^{opt}_{cb}(t)}{\hat{K}^{opt}_{cb}(t)};$

ea7) $\frac{\hat{X}^{\sigma}_{cb}(t)}{Y^d(t)}$, where $\sigma \equiv ec, +; ec, -$, and $X \equiv I,\ C,\ \hat{I},\ \hat{C}; G, \hat{G}$; $\frac{X^{\sigma}_{cb}(t)}{\hat{Y}^d(t)}$, X, σ as above;

ea7i) $\frac{\dot{X}^{\sigma}_{cb}(t)}{\hat{\dot{Y}}_{cb}(t)}$, $X,\ \sigma$ as above; $gr\left(\frac{X^{\sigma}_{cb}(t)}{\hat{Y}^d_{cb}(t)}\right)$, $X,\ \sigma$ as above;

ea8) $\frac{\hat{X}^{\sigma}_{cb}(t)}{Y^s(t)}, \frac{X^{\sigma}_{cb}(t)}{\hat{Y}^s_{cb}(t)}$, $X \equiv A,\ \hat{A},\ W,\ \hat{W},\ Z,\ \hat{Z}$, σ as above;

ea8i) $gr\left(\frac{X^{\sigma}_{cb}(t)}{Y^s(t)}\right), gr\left(\frac{X^{\sigma}_{cb}(t)}{\hat{Y}^s(t)}\right)$, $X \equiv \hat{A},\ \hat{W},\ \hat{Z}; \hat{W}^{FS}, \hat{W}^{PF}, \hat{Z}^{FS}$, $\hat{Z}^{PF}$, σ as above;

ea9) $\frac{\hat{Z}_{cb}(t)}{\hat{W}_{cb}(t)}, \frac{Z^{FS}(t)}{Z^{ec}(t)}, \frac{\hat{Z}^{FS}_{cb}}{\hat{Z}^{ec}_{cb}(t)}, \frac{W^{FS}(t)}{W^{ec}(t)}, \frac{N^{FS}(t)}{N^{em}(t)}, \frac{\left[\frac{Z^{FS}(t)}{N^{FS}(t)}\right]}{\left[\frac{Z^{ec}(t)}{N^{em}(t)}\right]}$, where N^{FS}, N^{em} denote employment in the financial sector and the whole economy, respectively;

Let U_i denote the degree of non – exploitation of factor "i" of production. Then, for $i \equiv K_{ph},\ K_f,\ K_h,\ K_s,\ K_{iar}, Technology$, in a given economy, we have:
$U^{ec}(t) = 6^{-1}\sum_{i=1}^{6} U_i(t)$ and $\hat{U}^{ec}_{cb}(t) = 6^{-1}\sum_{i=1}^{6} \hat{U}_{cb,i}(t)$;

ea10) $\widehat{eff}^{ec}_{cb}(t) = \left[U^{ec}(t) - \hat{U}^{ec}_{cb}(t)\right]^{+}$ denotes the positive impact of the use of CB – monetary tools.

Note 8.7. These propositions for indices measuring the results of the use of monetary policy tools use two kinds of symbols – those without and those with the sign " ˆ " placed overhead. The latter signifies quantities that need to be computed by the use of suitably chosen reasoning and formulas. Such formulas may be complex, since the palette of monetary tools is rich, and the choice of the weights of influence of particular tools will not be easy, either theoretically or empirically.

The above discussion does not exhaust the list of possible effects of the tools of CB – monetary policy. It could be interesting to consider, both theoretically and empirically, traces of the consequences of using CB – monetary tools with respect to qualitative and quantitative features of the following quantities and qualities:

Possible effects of the tools of monetary policies

ed1) differences between the supply of non-used capital goods and demand for them (e.g. unemployed people and job vacancies);

ed2) differences in the degrees of mobility of capital goods of type K_f, K_h, K_{ph}, K_{iar};

ed3) differences in the intensity of domestic or international innovations, entrepreneurship, competition;

ed4) differences in prolonging or shortening the duration of BC – phases;

ec1) changes (reinforcement or weakening) in the strength of trade unions;

ec2) changing (reinforcement or weakening) of built-in mechanisms of economic stabilization;

ec3) changing patterns of expectations among consumers, investors and politicians;

ec4) changing patterns for the setting of prices, wages, interest rates, trade, investments, production, services, state budgeting and credit;

ec5) changing the rate of increase or decrease in wages, profits and production gaps;

ec6) changes in the costs of inflation or deflation, rates of capacity utilization, inefficiency of monetary tools from the point of view of future generations of people, measured e.g. using the values of sacrificing ratios connected with them or welfare costs;

ec7) changes in the short, medium and long-term effects manifested in the discussed quantities and qualities;

ec8) changes in policies which favor or disfavor recent or future goods of type K_i, $i = ph,\ f,\ h,\ s,\ iar$;

ec9) changes in the neutrality or non-neutrality of monetary policy tools with respect to chosen effects;

ec10) changes in or invariance of the following trade – offs: $U \leftrightarrows \pi$, $\check{Y} \to \pi$, $\sigma^2_{\check{Y}} \to \sigma^2_{\pi}$, $cost\ (\pi) \to cost\left(\dot{\check{Y}}\right)$, $st(\check{Y}) \to st(\pi)$, $N^{em} \rightleftarrows \pi$, $\pi \to \dot{N}^{em}$, $\pi \to Vol^{trade}$, $\pi \to IR$, $\pi(t-j) \to lgN^{em}(t)$, $\pi \to Y^{sol}$, $\pi \to Z$, $R^{omo}_{cb} \to$ Inventories in the economy; $\check{Y}$ real value of Y; Y^{sol} denotes sold production;

ec11) changes in the robustness of the effects of monetary tools with respect to supply and demand, domestic or foreign shocks;

ec12) changes in monetary policy and tools due to radical economic or geopolitical shocks;

ec13) changes in endogenously and exogenously varying parts of the real and financial spheres of the economy due to interactions between inflation and personal income tax (PIT) or corporate income tax (CIT), inflation and capital gains tax (K_fIT) or intellectual property tax ($K_{iar}IT$) or interactions between $\pi = 0$ inflation and CPI, CIT, $K_fIT = 0$;

ec14) changes in the medium or long run patterns of C, I, Sav, accumulation of goods of type K_{ph}, K_f, K_h, K_s, K_{iar};

ec15) changes in the indices of credibility, uncertainty, accountability, time consistency (or inconsistency) of monetary policy tools used either in the form of rules or individual decisions;

ec16) changes in the roles of M0, M1 as measures of value, means of payments, stores of value, mediums of exchange, units of accounting;

ec17) changes in trends or fluctuations around trends of studied quantities due to changes in monetary tools;

ec18) changes in the correlation structure of the following quantities: π, $\check{Y}$, π^{ex}, $\check{Y} - \check{Y}_{cb}^{targ}$, $R_{cb} - R_{cb}^{targ}$, $R_{cb}\,(t+\tau) - R_{cb}(t+\tau)^{targ}$, U_i, $1 - U_i$, $N^{em,\,ex}$, $\check{Y}^{ex}$, Y^{ex}, E_{cb}, E^{ex}, R_{cb}^{δ}, $R_{cb}^{\rho\delta}$, R_{cb}^{omo}, $R_{cb}^{ex,\,omo}$, BOT, SB, BOP, U_i rate of unemployment of K_i; $targ$ corresponds to CB-target, SB denotes State Budget;

ec19) changes in the values of liquidity measures and their correlations with R_{cb}^{omo}, R_{cb}^{δ}, $R_{cb}^{\rho\delta}$, M^{res}, π, $\breve{\pi}$, $\dot{Y}$, $\dot{\check{Y}}$, U_i, Inv ≡Inventories; $\breve{\pi}$ – real inflation;

ec20) changes in the values of citizens' wealth, the structure of financial poverty, the domestic structure of citizens' liquidity preferences.

8.4. Final remarks

This chapter has given a brief presentation and discussion of the main theoretical and empirical problems related to the specification of the aims, tools, and effects of monetary policies conducted by central banks, which are dependent to varying degrees on the national government. The lists of these aims, tools and effects may be extended to include historical, current and possible future forms of monetary policies, both imagined or implemented in real life. This chapter has also considered some of the possible relations (logical, substantial) between (as individual groups) the aims, tools and effects, as well as between particular aims and tools, aims and effects, or tools and effects. From the point of view of modeling and forecasting, the most essential goal is to analyze the theoretical and empirical relations between empirically measurable tools and effects that can be empirically confirmed or falsified as models of chosen theories.

The above proposed empirical measures of monetary policy tools and effects may be used in the following extension of, f.ex., Taylorian type equation:

$$R_{cb}(t) = R^{eq}(t) + a_\pi \cdot (\hat{\Pi}_{cb}(t+\tau) - \hat{\Pi}^{eq}(t+\tau)) + a_y \cdot (\check{Y}(t) - \check{Y}_{cb}^{targ})+ \\ + a_E \cdot (E^w(t) - E^w_{cb}(t) \cdot (\hat{\Pi}(t+\tau) - \hat{\Pi}_{cb}(t+\tau)) + \varepsilon_{cb}(t),$$

$$\hat{\Pi}(t+\tau) - \hat{\Pi}_{cb}(t+\tau) = \\ = a_R \cdot (R_{cb}(t) - R^{eq}(t))(R_{cb}(t) - R_{ecb}(t)) + a_y \cdot (\check{Y}(t) - \check{Y}^{pot}(t)) + \varepsilon_\pi(t+\tau)$$

where the sign " $\hat{\cdot}$ " over the symbol denotes ex ante predictor, " $\check{\cdot}$ " denotes statistical correction enabling better comparability of statistical analysis results (called by economists transformation to the "real" values of economic data), $\check{Y}^{pot}$ denotes potential value of $\check{Y}$, $a_\pi, \ldots, a_y$ denote model parameters, $\varepsilon_{cb}, \varepsilon_\pi$ denote random specification errors.

The theoretical and their empirical counterparts of these relations may take linear or non-linear form, stochastic or deterministic form, one-equational or multi-equational form within the descriptive – behavioral approach without explicitly given optimality rule of CB or with the explicitly included optimality rule in the mathematical form. The first case received, f.ex., the forms analysed by J. Taylor, Ch. A. F. Goodhart, D. Henderson, W. J. McKibbin, J. C. Fuhrer, W. Milo. The second case embraces the model forms analysed, f.ex., by M. P. Giannoni, M. Woodford, J. J. Rotenberg, F. Smets, R. Wouters, M. Brzoza-Brzezina, M. Górajski, M. Ulrichs.

It is an open question, what would be the result of extensive empirical use of our empirically oriented measurements propositions for the analysed monetary policy instruments and their effects in the case of ad seriatim and feedback approaches relating instruments with effects, Frm the cognitive point of view, especially important would be finding the results of interaction between groups of instruments, and groups of effects when there is included time lagging.

Bibliography

Balch, Th. W. (1908). The Law of Oresme, Copernicus, and Gresham. *Proceedings of the American Philosophical Society*, Vol. 47, No. 188, pp. 18–29.

Barro, R., Fischer S. (1976). Recent Developments in Monetary Theory. *Journal of Monetary Economics*, *2*(2), pp. 133–167.

Böhm-Bawerk, E. (1891). *Capital and Interest. The Positive Theory of Capital (Vol. 2).* Macmillan, London.

Böhm-Bawerk, E. (1906). Capital and Interest Once More. I. capital vs. capital goods. *The Quarterly Journal of Economics*, *21*(1), pp. 1–21.

Breedon, F., Fisher, P. (1996). MO: Causes And Consequences. *The Manchester School*, *64*(4), pp. 371–387.

Brzoza-Brzezina, M. (2011). *Polska polityka pieniężna: badania teoretyczne i empiryczne.* Wydawnictwo CH Beck.

Clark, J. B. (1888). Capital and its Earnings . *Publications of the American Economic Association.* 4(1), pp. 39–63.

Clark, J. B. (1899). *The Distribution of Wealth.* Macmillan, New York.

Choudhry, M. (2001). *The Bond and Money Markets.* Butterworth-Heinemann, Oxford.

Copernicus, N. (1517). *N.C. Meditata.*

Copernicus, N. (1519). *Tractatus de monetis.*

De Long, B., Romer, C., & Romer, D. (1997). *Reducing Inflation: Motivation and Strategy.*

Eggertsson, G., Woodford, M. (2003). *The Zero Interest Rate Bound and Optimal Monetary Policy.* DP, Princeton University.

Elbourne, A., & de Haan, J. (2009). Modeling monetary policy transmission in acceding countries: Vector autoregression versus structural vector autoregression. *Emerging Markets Finance and Trade*, 45(2), pp. 4–20.

Fair, R. (1994). Is Monetary Policy Becoming Less Effective? *CFDP*, No. 1071, Yale University.

Fetter, F. W. (1932). Some Neglected Aspects of Gresham's Law. *The Quarterly Journal of Economics*, pp. 480–495.

Fischer, S. (1977). Long-Term Contracts, Rational Expectations, and the Optimal Money Supply Rule. *The Journal of Political Economy*, 85(1), pp. 191–205.

Fisher, I. (1930). *The Theory of Interest.* Macmillan, New York.

Friedman, B. (1988). Monetary Policy without Quantity Variables. *The American Economic Review*, pp. 1–17.

Friedman, B. (eds.) (1999). *Inflation, unemployment, and monetary policy*. MIT press.

Friedman, M. (1968). The Role of Monetary Policy. *The American Economic Review*, 58(1), pp. 1–17.

Froyen, R., Guender, A. (1998). Alternative Monetary Policy Rules for Small Open Economies. *DP* No. 9805, Dep. of Economics, University of Canterbury.

Górajski, M. (2017). Robust Monetary Policy in a Model of the Polish Economy: Is the Uncertainty Responsible for the Interest Rate Smoothing Effect? *Computational Economics*, pp. 1–28.

Górajski, M., & Ulrichs, M. (2016). Optymalne wrażliwe na ryzyko strategie polityki pieniężnej dla Polski. *Bank i Kredyt*, 47(1), pp. 1–32.

Hicks, J. (1935). A Suggestion for Simplifying the Theory of Money. *Economica*, *2*(5), pp. 1–19.

Jevons, W. S. (1875). *Money and the Mechanism of Exchange.*

Jevons, W. S. (1879). *The Theory of Political Economy.* Macmillan, London.

Keynes, J. M. (1936). *The General Theory of Interest, Employment and Money.* Macmillan, New York.

Menger, K. (1892). On the Origin of Money. *The Economic Journal*, *2*(6), pp. 239–255.

Milo, W. (red.). (2002). *Prognozowanie i symulacja.* Wydawnictwo Uniwersytetu Łódzkiego, Łódź.

Milo, W., Kozera, Z. et al. (2003). Inflacja cenowa finansowych dóbr inwestycyjnych. *Acta Universitatis Lodziensis. Folia Oeconomica*, Nr 166, pp. 101–117.

Milo, W., Szafrański, G., Wośko, Z., Malaczewski, M. (2006). *Rynki inwestycyjne a wzrost gospodarczy.* Wydawnictwo Uniwersytetu Łódzkiego, Łódź.

Milo, W., Malaczewski, M., Szafrański, G., Ulrichs, M., Wośko, Z. (2010). *Stabilność rynków kapitałowych, a wzrost gospodarczy.* Wyd. Nauk. PWN, Warszawa.

Phelps, E., Taylor, J. (1977). Stabilizing Powers of Monetary Policy Under Rational Expectations. *The Journal of Political Economy*, 85(1), pp. 163–190.

Poole, W. (1970). Optimal Choice of Monetary Policy Instruments in a Simple Stochastic Macro Model. *The Quarterly Journal of Economics*, T. 84, pp. 197–216.

Robertson, D. (1922). *Money. Cambridge Economic Handbook.* Nisbet and Co, London.

Romer, C. D., & Romer, D. H. (1997). *Reducing inflation: motivation and strategy.* University of Chicago Press.

Sargent, T., Wallace, N. (1973). The Stability of Models of Money and Growth with Perfect Foresight. *Econometrica: Journal of the Econometric Society*, Vol. 41, pp. 1043–1048.

Tobin, J. (1998). Monetary Policy. Recent Theory and Practice. *CFDP.* No. 1187, Yale University.

Wicksell, K. (1907). The Influence of the Rate of Interest on Prices. *The Economic Journal*, 17(66), pp. 213–220.

Wicksell, K. (1936). *Interest and Prices.* Macmillan, London.

Woelfel, Ch. (1994). *Encyclopedia of Banking and Finance.* Probus Publishing Company, Chicago.

Woodford, M. (2003). *Interest and Prices.* Princeton University Press, Princeton.

Chapter 9

Calculus of potentials in physics, mathematics and economics

Summary

This chapter presents elements of the calculus of potentials, which are used in physics, mathematics and economics. The last section presents some problems related to the calculation of theoretical economic potentials based on R. Solow's model.

Keywords: potential, potential energy, potential force, field potential, logarithmic potential, potential of a kernel, Solow's model, economic growth potential.

9.1. Introduction

"Potentia" in Latin[1] denotes: possibility, contingency, feasibility, compatibility, capability, attainability, accessibility, permissibility, conceivability, natural ability, degree of power. These words most often mean that something is, mental, not existing in factual being, but may, under appropriate conditions, come into existence. According to Aristotle's school, potential

[1] In modern English the word "potentia" has been replaced by the Latin adjective "potential(is)". Unless, the text refers directly to the use of this concept in classical times, the modern word "potential" will be used. This English word is more flexible, since it can be used as both an adjective and a noun.

(in Greek: $\delta\upsilon\nu\alpha\mu\iota\varsigma$) means – what is possible to happen, but has not yet occurred, or, what can come into existence, but does not yet exist.

In physics, e.g. potential is a quantity characterizing fields of the forces of gravity, electricity, inertia, pressure, resistance, friction and elasticity, as well as different forms of energy, motion, impetus, speed and acceleration.

In mathematics, potential means the solution of an appropriately posed mathematical problem involving partial differential equations. J. Lagrange (1773, Oeuvres VI, p. 335) was the first to introduce this concept, which was extended by Laplace (1782), Poisson (1813), Green (1828), d'Alambert (1743), Gauss (1840), Dirichlet (1898), Neumann F. (1887) and Robin (1886).

In Section 2 there is a brief discussion of the concept of potential, especially from the point of view of physics.

Section 3 contains descriptions of several approaches to this concept from the point of view of mathematical disciplines.

Section 4 includes a brief discussion of problems connected with measuring economic growth potential.

The most difficult, especially from a practical point of view, is to establish basic principles for measuring potentials, as well as fundamental principles for the calculus of potentials that can be used in both the natural and social sciences, in particular in economics.

9.2. The measurement and calculus of potentials in physics

The classical era gave us a very broad scope of meanings of potentia. In Aristotle's school, potentia, denoted here by Π, had the following meanings:

- p1) source or cause of motion, change, rest that is attributed to a given thing, person, set of things or set of persons;
- p2) ability, capacity of a person, thing to be subject, liable, dependent to motion effected by an exogenous cause or cause per se;
- p3) possibility, capacity of person, thing to realize a given goal according to one's own or someone else's will, need, necessity or enforcement;
- p4) states of things or their surroundings to be robust to disturbances;

p5) capacity to develop, fall, create, auto-destruct, or the principle of transition from development to fall;
p6) possession of states of capacity by person(s), thing(s), system(s).

These meanings are so general that they may refer to many modern fields of science, such as physics, chemistry, biology, engineering and economics.

In the context of modern formulations of Π, references to these fields would be quite interesting in content and form. Below, we focus our attention on two kinds of meanings of potential, i.e. those that are used in physics and mathematics.

Let us start from the meanings of this word in physics. They can be found in many books and articles published in well-known journals and book series devoted to physics. Detailed comparisons of the different meanings of potential used in physics are very important for historians of scientific development, although they are time consuming, due to the vast international literature in many languages, varying notation of both the appropriate variables and parameters in the models which are used as the basis for calculating the magnitude of potentials. Most often, the word "potential" is found together with such words as: energy, force, field, work, velocity, acceleration, momentum. Thus[2] we can find: potential energy E_p, potential force $\mathbf{F}_p$, potential field ϕ_p, potential work A_p, potential velocity $\dot{\mathbf{R}}_p$, potential acceleration $\ddot{\mathbf{R}}_p$, potential momentum M_p. Here, the word potential is used as an adjective.

However, regardless of whether this word is used as an adjective or as a noun, it can be seen that the designates of the concept of potential are attributed to many categories of physical quantities. This means that we can define potential Π as follows:

d1) $\Pi_E\left(t,\ \mathbf{R}(t),\ \dot{\mathbf{R}}(t)\right) \equiv E_p\left(t,\ \mathbf{R}(t),\ \dot{\mathbf{R}}(t)\right) = \varphi_1\left(t,\ \dot{\mathbf{R}}(t)\right)$
d2) $\Pi_{\dot{\mathbf{R}}}\left(t,\ \mathbf{R}(t),\ \dot{\mathbf{R}}(t)\right) \equiv \varphi_2\left(\dot{\mathbf{R}}(t), t\right)$
d3) $\Pi_{\mathbf{F}}\left(t,\ \mathbf{R}(t),\ \ddot{\mathbf{R}}(t)\right) \equiv \varphi_3\left(t, \ddot{\mathbf{R}}(t)\right)$
d4) $\Pi_\Phi(t,\ \Phi(t)) \equiv \varphi_4(t,\ \Phi(t))$
d5) $\Pi_A\left(t,\ \dot{\mathbf{R}}(t), A(t)\right) \equiv \varphi_5(t, A(t))$
d6) $\Pi_M(t, M(t)) \equiv \varphi_6(t,\ M(t))$.

In the above notation, t denotes time, $\mathbf{R}(t)$ denotes the vector of the positional coordinates of a given point or group of points or a whole physical

[2] Perhaps it would be more instructive to say, e.g. the potential of energy or energy's potential E_p, and similarly for other kinds of potential.

body represented geometrically by a set of points, e.g. $\mathbf{R}(t) \in \mathbf{R}^3$, and $\mathbf{R}^3$ denote three dimensional Euclidean space. The symbols $\{\varphi_i\}_1^6$ denote appropriate functions known from physics, e.g.

$$\varphi_1\left(t, \dot{R}(t)\right) \equiv m \cdot \left(\dot{R}(t)\right)^2, \ \dot{R}(t) = \|\mathbf{R}(t)\| \ .$$

In the case of gravitational fields, many authors also underline that the potential of energy can be defined as $\Pi_E \equiv -A$, where $(-A)$ comes from the German word "Arbeit". This potential is the scalar quantity of work done by a body due to its kinetic force $\mathbf{F}_k$, when this motion is exchanged for a growing potential force $\mathbf{F}_p$ until it reaches $\mathbf{F}_p^{max}$ when the body comes to rest. The potential energy E_p of the physical system of bodies S can be interpreted as the accumulated kinetic energy which, under some appropriate conditions, may by released or regained as kinetic energy E_k.

The conversion of $E_p \to E_k$ or $E_k \to E_p$ is due to the replacement of $\mathbf{F}_p$ by $\mathbf{F}_k$ or $\mathbf{F}_k$ by $\mathbf{F}_p$, respectively, although it must be said that, physically, $\mathbf{F}_p$ in gravitational fields is the kinetic force of gravitation of the (first) body that attracts more strongly than another (second) body that is "raised, shifted" by a third body[3]. The source of $E_p, \ \mathbf{F}_p, \ A_p$ could be in the internal physical constitution of the system S, i.e. in the physical and chemical features of the matter belonging to S that are called the elasticity, compressibility or strength of that material. These properties determine the reaction of a material to the external force $\mathbf{F}_{k,\ ex}$ based on an internal field of forces acting within S.

It is known that the value $\Pi \equiv E_p = -A$ depends on the value of the position vector $\mathbf{r} = (x, y, z) \in R^3$ of S. Hence, our real scalar functions Π, E_p, A are in fact functions $\Pi(\mathbf{r}), \ E_p(\mathbf{r}), \ A \equiv A_{10}(\mathbf{r}_1, \ \mathbf{r}_0) = \Pi(\mathbf{r}_1) - \Pi(\mathbf{r}_0)$, $\mathbf{r}_1, \ \mathbf{r}_0 \in B^3$, and $A_{01} = -A_{10}$, $A(\mathbf{s}) = \mathbf{F}_p \cdot \mathbf{s}$, $\mathbf{s} = \overline{\mathbf{r}_0\mathbf{r}_1}$, where $\mathbf{F}_p$ is a potential force vector.

The most fundamental and computationally useful definition used in the calculus of potentials by physicists and mathematicians is

$$\mathbf{F}_p(\mathbf{r}) \equiv (\pm)\,\dot{\Pi}(\mathbf{r}), \dot{\Pi}(\mathbf{r}) \equiv \nabla\Pi(\mathbf{r}) \equiv (\partial\Pi(\mathbf{r}) / \partial(\mathbf{r})), \tag{9.1}$$

where $\mathbf{F}_p(\mathbf{r})$ is here interpreted as the rate of change of potential Π at the position (place) $\mathbf{r}$ of the body or system considered.

The convention defined by (9.1) means that given the analytical form of $\Pi(\mathbf{r})$, we may calculate the analytical form of the potential force function

[3] If this third body, apart from raising the second body, gives constant support to it, then the conversion of $\mathbf{F}_k \to \mathbf{F}_p \to \mathbf{F}_k$ and $E_k \to E_p \to E_k$ may take place only after the support supplying the anti-gravitational force that was keeping the second body in place ceases to act.

$\mathbf{F}_p(\mathbf{r})$ or, reversing the problem, given the analytical form of $\mathbf{F}_p(\mathbf{r})$, we may calculate the analytical form of $\Pi(\mathbf{r})$.

To illustrate such a calculation and the interpretation of results, we first consider elasticity fields and their potentials.

Example 1. Assume that elastic potential force $\mathbf{F}_{el}$ satisfies

$$\Pi(\mathbf{F}_{el}) \equiv E_p(\mathbf{F}_{el}) = 2^{-1}k\tilde{x}^2 + c. \tag{9.2}$$

Then, given

.) $\mathbf{r}(x, y, z)$ is such that $x_0 = 0$, $x_1 = x$, $\tilde{x} = x_1 - x_0$, and c, k are constants,

we have

..) $\mathbf{F}_{el}(r)$ is an elasticity force with $y, z = 0$, and x as given in (.),

...) for $\partial\Pi/\partial y = \partial\Pi/\partial z = 0 \rightarrow \partial\Pi/\partial\mathbf{r} = (k\tilde{x}, 0, 0) = \mathbf{F}_{p,el}(\mathbf{r})$, (9.3)

where $\tilde{x}$, is the velocity (a function of t).

In the case of gravitational potential, we have

$$\Pi(\mathbf{F}_{p,gr}) \equiv E_p(\mathbf{F}_{p,gr}) = -mgz + c, \tag{9.4}$$

where

.) $x, y = 0$ in $\mathbf{r} = (0, 0, z)$

..) m, c are constant in time; m is the mass of an object,

...) $\mathbf{F}_{p,gr}(\mathbf{r})$ is the gravitational force at $\mathbf{r}$, $g = \|\mathbf{g}\|$ is the rate of acceleration for the free fall of any body in a given region. We obtain the following implication:

.v) for $\partial\Pi/\partial x = \partial\Pi/\partial y = 0 \rightarrow \partial\Pi/\partial\mathbf{r} = (0, 0, -mg) = \mathbf{F}_{p,gr}(\mathbf{r})$. (9.5)

Note 9.1. In (9.2), x_0 may denote, e.g. the S-equilibrium value, S-stability value, S-robustness value, or S-fragility value. For these, possibly different values of x_0, we may obtain, very practically important values of Π, E_p, A that measure potential, potential energy or the work of a system of bodies S originally not in these specific states x_0 of S. For $x_0 = 0$, the proposed interpretations are not of practical use.

Note 9.2. If we replace (9.2) by

$$\Pi(F_{el}) \equiv E_p(F_{el}) = 2^{-1}\left(k\tilde{x}^2 + \kappa\tilde{x}\right) + c, \kappa \neq 0, \tag{9.6}$$

then

$$\partial\Pi/\partial y = \partial\Pi/\partial z = 0 \Rightarrow \mathbf{F}_{p,el}(\mathbf{r}) \equiv \partial\Pi/\partial r = (k\tilde{x} + \kappa, 0, 0)\,. \tag{9.7}$$

Thus the non-zero coordinate of the elasticity force vector will have an additional additive constant, κ.

Replacing (9.4) by

$$\Pi(\mathbf{F}_{p,gr}) \equiv E_p(\mathbf{F}_{p,gr}) = -mgz + 2^{-1}m\omega^2\varrho z + c, \tag{9.8}$$

it is easy to find that

$$\partial\Pi/\partial x = \partial\Pi/\partial y = 0 \Rightarrow \mathbf{F}_{p,gr} \equiv \partial\Pi/\partial\mathbf{r} = (0, 0, -mg + 2^{-1}m\omega^2 z), \tag{9.9}$$

where $2^{-1}m\omega^2 z$ is the part of $\mathbf{F}_{p,gr}$ that reflects a Coriolis type effect.

Note 9.3. For the potential of a gravitational field

$$\Pi(\mathbf{F}_{p,gr}) = m^{-1}E_p(r) = -m^{-1}gr^{-1}mM = -gr^{-1}M,$$

where M is a mass that attracts a smaller mass m, $r = dist\,(M, m)$, g is the acceleration of the motion of m. Hence, we obtain

$$\mathbf{F}_{p,gr} = \partial\Pi/\partial\mathbf{r} = (0, 0, -gM)\,. \tag{9.10}$$

Note 9.4. Forces with any combination of arguments t, $\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}$, e.g. the forces $\mathbf{F}(t,\ \mathbf{r}, \dot{\mathbf{r}})$, $\mathbf{F}(t,\ \mathbf{r}, \ddot{\mathbf{r}})$, $\mathbf{F}(t,\ \mathbf{r}, \dot{\mathbf{r}},\ \ddot{\mathbf{r}})$ are not potentials in the sense of the convention (9.1), but such a convention may be relaxed.

Note 9.5. If $\mathbf{F}_{el}(\mathbf{r}) = -\,k^2\widetilde{\mathbf{r}}$, where $\widetilde{\mathbf{r}} = \mathbf{r} - \mathbf{r}^*$ and $\mathbf{r}^*$ is the equilibrium position, then

$$\Pi(\mathbf{F}_{p,el}(\mathbf{r})) = 2^{-1}k^2r^2 + c. \tag{9.11}$$

Note 9.6. For the logarithmic potential, $\Pi(\mathbf{F}_{gr}) = -E_{p,w}$,

$$\Pi(\mathbf{F}_{p,gr}) = 2c\sigma ln\,(\varrho(r)) + \Upsilon, \varrho = \left(x^2 + y^2 + z^2\right)^{1/2}. \tag{9.12}$$

The potential force is given by

$$\mathbf{F}_{p,gr} = \partial\Pi/\partial\mathbf{r} = 2c\delta\left(x^2 + y^2 + z^2\right)^{-1} \cdot \mathbf{r}, \mathbf{r} = \left(x^2, y^2, z^2\right)^T. \tag{9.13}$$

Note 9.7. It is known that if

$$\mathbf{F} = \mathbf{F}_p(\mathbf{r}) \Longrightarrow rot\,(\mathbf{F}_p(\mathbf{r})) = 0, \mathbf{F}_p(\mathbf{r}) = -\nabla\Pi(\mathbf{r}) \tag{9.14}$$

and the field potential force is a no-whirling field.

Note 9.8. In all cases, if

$$E_p(\mathbf{r})=\Pi(\mathbf{r})=0 \Longrightarrow \mathbf{r}=\mathbf{r}^*=\mathbf{r}^{eq} \text{ and } \ddot{\mathbf{r}}=\mathbf{0}.$$

Moreover, for $\Pi(\mathbf{r}^*)=E_p(\mathbf{r}^*)$, the values of Π, and E_p are minimal, togeether with the property that $\mathbf{r}^*$ is a stable equilibrium, i.e. for small changes in $\partial\mathbf{r}$, $\partial\dot{\mathbf{r}}$ the system returns to the equilibrium position $\mathbf{r}^*$, which follows from Dirichlet's theorem.

There are also other formulas for potentials proposed within, e.g. thermodynamics, quantum mechanics and Lagrange-Hamilton mechanics. These are not presented, due to constraints on the length of the book and it is not necessary to complicate the description of the foundations of the calculus of potentials.

However, It would be instructive to show the consequences of departures from the definition of $E_p \equiv \Pi = 2^{-1}k\tilde{x}^2 + c$. Firstly, suppose

$$\Pi(\mathbf{F}_{el}) \equiv E_p(\mathbf{F}_{el})=2^{-1}k\tilde{x}^2+2^{-2}k^2\tilde{x}^4, \tag{9.15}$$

then for

$$\partial\Pi/\partial y = \partial\Pi/\partial z = 0 \Rightarrow \mathbf{F}_{p,el}(\mathbf{r}) = \frac{\partial\Pi}{\partial\mathbf{r}} = \left(k\tilde{x}+k^2\tilde{x}^3,0,0\right).$$

Secondly,

$$\Pi(\mathbf{F}_{el}) \equiv E_p(\mathbf{F}_{el})=2^{-1}k_1\tilde{x}^2+k_2lg|\tilde{x}|^{-1}+c \Rightarrow$$

$$\mathbf{F}_{p,el}(\mathbf{r}) = \partial\Pi/\partial\mathbf{r}=(k_1\tilde{x}+k_2\tilde{x},0,0)=((k_1+k_2)\,\tilde{x},0,0). \tag{9.16}$$

Thirdly,

$$\Pi(\mathbf{F}_{el}) \equiv E_p(\mathbf{F}_{el})=k\tilde{x}^2lg|\tilde{x}|^{-1}+c \Rightarrow$$

$$\mathbf{F}_{p,el}(\mathbf{r}) = \partial\Pi/\partial\mathbf{r}=\left[k\tilde{x}\cdot\left(2lg|\tilde{x}|^{-1}+\tilde{x}^2\right),0,0\right]. \tag{9.17}$$

Note 9.9. The consequences of additive shifts in the shape of potential functions are the corresponding additive shifts in the non-zero coordinate of the vector of potential force. In the case of such shifts to (9.16), when k_2 is an arbitrarily small real number, the small shift in the potential forces causes a corresponding shift of magnitude $k_2lg|\tilde{x}|^{-1}$ in the potential energy.

Note 9.10. Additive – product compositions of the non—zero coordinate of the force vector cause less complex products from composition in the potential function.

Note 9.11. It is an open question as to whether such shifts in the values of the force functions change their extreme values or cause changes in the extreme values of the energy potential (with respect to the state of extreme values from (9.2)).

The above discussion refers to physical situations in which the energy or force of kinesis (motion) is appropriately determined by the velocity or acceleration of a body in motion and the mass of the body taken together in the form of a product. This approach has brought useful results in solving many problems from engineering, chemistry or biophysics. However, in this approach, there are such outside factors as: time as argument of velocity function, variability of velocity and acceleration in time, mass that varies over time, transfer of energy from a body to another body with greater potential or vice versa, hysteresis of movement effects, the potential of systems of bodies with different degrees of interactions between the bodies, calculus of potentials without Newtonian mechanics, resonant systems with inertia, gravitational, elasticity and kinetic forces.

There are views (see e.g. I. Bouhalov (2009)) that in the XXc the important problem in physics of unifying general relativity theory (GRT) and quantum fields was not solved. The main obstacle to such a unification is the universal interactive force called gravity which all kinds of matter participate in.

Recently, even Einstein's Ether (EE) hypothesis has been studied, despite his rejection of the idea of ether. It can be seen that the idea of a vacuum in QFT (quantum field theory) has replaced the role that ether previously played. Therefore, the adherents of GRT are searching for the limits of the use of EE and trying to design experiments that would validate or reject these claims.

Thus, if one assumes that a vacuum is a specific form of matter (so it is not "nothing", but is "something"), then it can be interpreted ontologically as the basis of matter. If we do not assume that it is a form of matter, then the concept of a vacuum will only play the part of an abstract idea, however pleasing, without any substantial content. Attempts to give a quantum interpretation of inertia lead to inconsistencies. One of them is as follows: The equivalency principle of GRT leads to the statement that, locally, inertia is a gravitational field, or in QFT terms, a virtual gravitational field. However, it is difficult to give the status of a real physical object to such a field. If such a quantum field existed, its existence should be observed, since according to GRT the gravitational force field is formed by geometric coordinates, not by physical bodies, as is a standard gravitational field. These considerations show how difficult it is to introduce an ontologically neutral language (ONL) in specific natural sciences. These kinds of difficulties can be seen e.g. from T. Hailperin (2001).

Similar kinds of difficulties are discussed in the papers of Y. Aharonov and D. Bohm (1959) and M. Bunge (2014). In the second paper, it is shown that a field of electromagnetic intensities is given by values of vector potentials, though such potentials have richer representations than these intensities, since they also describe the effects of this field on charged matter, and these two notions have different senses. Thus, the theories in the field of physics disregard reference as a component of meaning, so the usual logical theories of meaning are of no use here. Therefore, when a researcher is not making any explicit reference to physical things, he is only studying the properties of mathematical objects. However, it must be stressed that a principal reason as to why the notion of potential energy was introduced into physics (both classical and quantum) is the "book-keeping" principle of the conservation of energy, i.e. that the total energy of a physical system is constant in time and equals the sum of kinetic and potential energy. All the three quantities to be calculated, i.e. E_k, E_p, c are, however, given in terms of a convention and only known in a thought experiment where the identity $E_k + E_p = c$ is satisfied. This means, that for any computed E_k there are as many values of E_p as there are values of c that we can assume. However, kinetic energy is given by $E_k = 2^{-1}m\dot{r}^2$ for any velocity $\dot{r}\,(t)$. Hence, the maximal possible (therefore in a sense potential) kinetic energy, i.e. $E_k^{max} = m\dot{r}_c^2$, where $\dot{r}_c$ is the speed of light in a vacuum, and this speed is Einstein's physical constant, treated as the maximal possible speed in the universe.

9.3. The measurement and calculus of potentials in mathematics

9.3.1. Classical results

O. D. Kellog (1929, p. 52) noted that the concept of "a field of gravitational force" (today called a potential field), together with the appropriate function describing it, appeared as early as J. Lagrange (1773). Such a function was referred to as a "potential function" by G. Green (1828) and later as "potentia (l)" by C. Gauss (1840).

Due to Laplace, Gauss, and others, it was confirmed that the method of potentials is useful, in solving problems from gravitational theory and other areas of mathematical physics. Such problems were formulated mathematically by Laplace, Green, Dirichlet, Neumann, Lapunov, Steklov, Kellog, Zaremba, Wilkosz, Brelot, Wiener, Bauer, Bouligand, Doob, Cartan, Calderon, Riesz, Stein, Choquet and Hervé.

Let us recall, see Kellog (1929, p. 48–54), that a particle with mass m will move from point $P(t_0) \equiv P_0$ at time t, according to Newton's law of motion:

$$m\frac{d^2x}{dt^2} = \lambda mX, m\frac{d^2y}{dt^2} = \lambda mY, m\frac{d^2z}{dt^2} = \lambda mZ, \tag{9.18}$$

where λ is a constant, so adding these equations together gives

$$m\left[\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} + \frac{d^2z}{dt^2}\right] = \lambda m\left[X + Y + Z\right].$$

The use of the convention $E_k = 2^{-1}m\dot{r}^2$, multiplication by dx/dt, dy/dt, dz/dt and integration of both sides leads to

$$E_k(t) - E_k(t_0) = \lambda m \int_{t_0}^{t} \left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right)dt =$$

$$\lambda m \int_{P_0}^{P} (Xdx + Ydy + Zdz) \Rightarrow$$

$$E_k - E_{k,0} = \lambda m A(P, P_0), \ X = \partial A/\partial x, \ Y = \partial A/\partial y, \ Z = \partial A/\partial z. \tag{9.19}$$

For conservative vector fields of forces or lamellars and fixed point P_0, the potential energy is given by $E_p = -\lambda m A(P, P_0)$, $A(P, P_0) = \int_{P_0}^{P} F cos(\theta)\, ds$.

In a sense, A is a force function that sometimes coincides with $(-\Pi)$ for a vector field, or in other words $\Pi \equiv E_p = -\lambda m A$.

For abstract fields, Π is an antigradient of (X, Y, Z) and corresponds to $(-E_p)$ and A. Hence, $(X, Y, Z) = \nabla \Pi$.

For repelling forces, we have $(X, Y, Z) = -\nabla \ \Pi$, so $\Pi = E_p$ and is thus the negative of the force function.

There are many possible ways of defining potentials for a point, a line L, surface S or body V. For instance, the last three kinds of potentials can be defined as

$$\Pi_L \equiv \int_L \lambda r^{-1} dL, \ \Pi_S \equiv \iint_S \lambda r^{-1} dS, \Pi_V \equiv \iiint_V r^{-1} \kappa dV, \qquad (9.20)$$

or the corresponding logarithmic ones

$$\Pi_{LL} \equiv \int_L \lambda lg \left(r^{-1}\right) dL$$

$$\Pi_{LS} \equiv \iint_S \sigma lg \left(r^{-1}\right) dS, \Pi_{LV} \equiv \iiint \kappa lg \left(r^{-1}\right) dV, \qquad (9.21)$$

where r^{-1} is, in the theory of Newtonian potentials, the potential of a unit positive particle, point charge or magnetic pole (recall that the force generating a potential can be attracting or repulsive).

Despite the fact that Lagrange posed many new problems in the field of expressing problems from physics in the language of differential equations, it was Laplace who introduced the beautiful and very useful idea of a harmonic function.

Definition 1. The function $u(\mathbf{X})$ is harmonic in the region $\Omega \in R_n$ of the space R_n if $u \in C^2(\Omega)$ and u fulfills the Laplace equation $\Delta u(\mathbf{X}) = 0$, where $\Delta \equiv \sum_{j=1}^{n} D_j^2$, $D_j \equiv \partial / \partial x_j$, and Δ is the Laplace differential operator or Laplacian.

Usually, in practice, it suffices to consider $n = 3$ and $\Omega \subset R_3$, e.g. Ω is a set of electrical charges with density $\varrho(\mathbf{X})$. The charges constitute an electrostatic vector field with potential Π.

Poisson generalized Laplace's solution $\hat{u}$ of the problem $\Delta u(\mathbf{X}) = 0$, $\mathbf{X} \in R_n$ to finding the solution of the equation $\triangle u(\mathbf{X}) = f(\mathbf{X})$, $\mathbf{X} \in R_n$, where f is known (given).

The solutions of the Laplace and Poisson problems do not always exist, i.e. there are no appropriate harmonic functions u and $\hat{u}$. It would be instructive to recall, see, e.g. H. Marcinkowska (1972, Ch. 4), examples of harmonic functions that can be found in the literature. These are as follows:

Example 1. $u(\mathbf{X}) = |\mathbf{X}|^{2-n}$ is harmonic in $\Omega \backslash \{0\}$; for instance when $n = 3$, $u(\mathbf{X}) \equiv \Pi(\mathbf{X}) = |\mathbf{X}|^{-1}$ is the potential produced by a unit electric charge placed at the point $(0, 0)$;

Example 2. $u(\mathbf{X}) = lg|\mathbf{X}|^{-1}$, $\mathbf{X} \in R_2$; u is harmonic in $R_2 \Rightarrow \Pi(\mathbf{X}) = 2\mu lg|\mathbf{X}|^{-1}$ so Π is a logarithmic potential.

The following are elementary solutions of the Laplace equation:

$$E(\mathbf{X}) \equiv \hat{u}_L(\mathbf{X}) = (2\Pi)^{-1} lg\,|\mathbf{X}|\,, \text{ if } n = 2,\ \Omega = R^2,$$

$$E(\mathbf{X}) \equiv \hat{u}_L(\mathbf{X}) = -[(4\Pi)\,|\mathbf{X}|]^{-1}, \text{ when } \theta_3 = 1, n = 3,$$

$$E(\mathbf{X}) \equiv \hat{u}_L(\mathbf{X}) = -\left[(n-2)\,\theta_n |\mathbf{X}|^{n-2}\right]^{-1}, \text{ for } n > 2,$$

$$E(\mathbf{X}) \equiv \hat{u}_L(\mathbf{X}) = \alpha lg|\mathbf{X}-\mathbf{y}|^2.$$

It is known that for harmonic function u in Ω, $u : \Omega \to R$, we have $u(\mathbf{X}) = \int_{\partial\Omega} u(\mathbf{y}) \frac{\partial E(\mathbf{X}-\mathbf{y})}{\partial_y \overrightarrow{n}} d\sigma_y - \int_{\partial\Omega} \frac{\partial u}{\partial_y \overrightarrow{n}} E(\mathbf{X}-\mathbf{y})\, d\sigma_y$, where $\overrightarrow{n}$ is a directed vector normal to the exterior of Ω, and $\partial\Omega$ is the boundary of Ω. For $\Omega = \mathrm{R}^2$ and $x = 0$ in $\triangle u = 0$, we also have that $u \in C_0^\infty(\Omega)$, and

$$u(0) = \int_\Omega E(y) \triangle u(y)\, dy.$$

It would be useful to introduce, see Th. 4.1.2 in Marcinkowska's monography (1972), the following statement:

Theorem. Let Ω be a bounded set where $\partial\Omega$ is from the class C^1. Each $u \in C^2\left(\overline{\Omega}\right)$ can be represented by the sum of the following Newtonian potentials:

$$u(\mathbf{X}) = \Pi_{vol}(\mathbf{X}) + \Pi_{\text{1layer}}(\mathbf{X}) + \Pi_{\text{2layers}}(\mathbf{X})\,. \tag{9.22}$$

In the case where u is harmonic in Ω, it follows that

$$u(\mathbf{X}) = \Pi_{\text{1layer}}(\mathbf{X}) + \Pi_{\text{2layers}}(\mathbf{X})\,, \tag{9.23}$$

and u is then also analytic in $\Omega \subset R_4$.

It should be recalled that for $n > 2$

$$\Pi_{vol}(\mathbf{X}_c) = \int_\Omega \varrho(\mathbf{X}) E(\mathbf{X}_c - \mathbf{X})\, d\mathbf{X}, \mathbf{X}_c \notin \Omega, \tag{9.24}$$

where ϱ is measurable, finite and determined in Ω with diam $(\Omega) < \infty$.

For a surface $\sum$ from the class C^1 and functions μ, ν which are integrable on $\sum$, the one layer potential for $\mathbf{X}_C \notin \sum$ is defined to be

$$\Pi_{1 \text{ layer}}(\mathbf{X}_C) = \int_{\sum} \mu(\mathbf{X}) E(\mathbf{X}_C - \mathbf{X}) d\sigma_{\mathbf{X}}, \tag{9.25}$$

where $d\sigma_{\mathbf{X}}$ is a natural measure of surface area on $\sum$, and the two layer potential is of the form

$$\Pi_{2 \text{ layers}}(\mathbf{X}_C) = \int_{\sum} \nu(\mathbf{X}) \frac{\partial E(\mathbf{X}_C - \mathbf{X})}{\partial_y \overrightarrow{n}} d\sigma_{\mathbf{X}}, \mathbf{X}_C \notin \sum . \tag{9.26}$$

In electrostatics, when $n = 3$ and $\mathbf{X}_C \notin \sum \subset R_3$, the integral expression $-4\pi\Pi_{vol}(\mathbf{X}_C)$ represents the value of the electrostatic potential at $\mathbf{X}$, where Π_{vol} is generated in a vacuum by electric charges distributed over Ω with density $\varrho_\Omega \equiv \varrho$.

The integral $-4\pi\Pi_{1\text{layer}}(\mathbf{X}_C)$ represents, in turn, values of one layer electrostatic potential at the point $\mathbf{X} \notin \sum$, where the values are generated in a vacuum by electrostatic charges distributed over $\sum$ with density μ. When there are two layers, it is known that the integral $-4\pi\Pi_{2\text{layers}}(\mathbf{X}_C)$ represents the value of the two layer electrostatic potential at the point $\mathbf{X}_C \notin \sum$ generated in a vacuum by electric charges in $\sum$ distributed by electric dipoles with an axis normal to $\sum$, where ν is the density of the dipole moment.

It should be recalled that $\Pi_{1\text{layer}}$ denotes the logarithmic potential of a single layer, and $\Pi_{2\text{layers}}$ denotes the logarithmic potential of two layers.

There are many forms of boundary value problems (BVP). A brief presentation that can be recommended is given by A. I. Prilenko, J. D. Sołomentzev (1984, T4, pp. 523–531). We will use the following notation:

S – a smooth closed surface with smoothness of order $(n-1)$, $S \subset R_n$, such that for $n \geq 2$, S encloses the finite region $\Omega = \Omega^+$, $\partial\Omega = S$ with $\Omega^- = R_n \backslash (\Omega^+ \cup S)$ denoting the non-finite external region; $E(\mathbf{X}_C, \mathbf{X}) = E(|\mathbf{X}_C - \mathbf{X}|)$ – the solution of the Laplace equation $\triangle u = 0$, where

1. $|\mathbf{X}_C - \mathbf{X}| = \left[\sum_{i=1}^n (x_{C,i} - x_i)^2\right]^{1/2}$ is the Euclidean distance between $\mathbf{X}_C$ and $\mathbf{X}$ in R_n,
2. $E(\mathbf{X}_C, \mathbf{X}) = \begin{cases} \left[\theta_n \cdot (n-2) \cdot |\mathbf{X}_C - \mathbf{X}|^{n-2}\right]^{-1} & \text{if } n \geq 3 \\ \left[(2\Pi)^{-1} ln|\mathbf{X}_C - \mathbf{X}|^{-1}\right] & \text{if } n = 2 \end{cases}$;
3. $\theta_n = (2\Pi)^{n/2}\mathbf{\Gamma}(n/2)$, where $\mathbf{\Gamma}$ denotes the gamma function;

Let us recall that the absolutely integrable functions ϱ, μ, ν are called the densities of the corresponding potentials Π_{vol}, $\Pi_{1\text{layer}}$, $\Pi_{2\text{layers}}$.

If $\varrho \in L_1(\Omega)$, then $\Pi_{vol}(\mathbf{X}_c)$ is harmonic for $\mathbf{X}_c \in \Omega^-$ and the volume potential $\Pi_{vol}(\mathbf{X}_c)$ is summable in Ω^+. Moreover, if $\varrho \in L_p(\Omega)$ and $1 \leq p \leq n/2$, then $\Pi_{vol} \in L_p(R_n)$ with $\frac{1}{p} + \frac{1}{q} = 1$, $1 < q < np/(n-2p)$.

In the case where μ is summable and bounded, $\Pi_{1\text{layer}} \equiv \Pi_{1l} \in C^{(0,\lambda)}$, $\lambda(0,1)$. For $\nu \in C^{(l,\alpha)}(S)$, $S \in C^{(k+1,\alpha)}$, $0 < \alpha < 1$, l, k integers , $0 \leq l \leq k+1$, then $\Pi_{2\text{layers}} \equiv \Pi_{2l} \in C^{(l,\alpha)}$ in Ω^+ or Ω^-.

When solving BVP (boundary value problems) and deriving properties of their solutions, the following Green's formula is used:

$$q(\mathbf{X}_c)\Psi(\mathbf{X}_c) =$$

$$-\int_\Omega \Delta\Psi(\mathbf{X}) E(\mathbf{X}_c, \mathbf{X})\, d\mathbf{X} +$$

$$+\int_{\mathbf{S}} \left[\frac{\partial\Psi(\mathbf{X})}{\partial\overrightarrow{n}_x} E(\mathbf{X}_c, \mathbf{X}) - \Psi(\mathbf{X}) \frac{\partial E(\mathbf{X}_c, \mathbf{X})}{\partial\overrightarrow{n}_x}\right] dS(\mathbf{X}), \tag{9.27}$$

where $\Psi \in C^2(\Omega \cup S)$, S is a smooth surface.

The function Ψ is useful in the following calculus $\varrho(\mathbf{X}) = -\Delta\Psi(\mathbf{X})$, $\mu(\mathbf{X}) = \varrho\Psi(\mathbf{X})/\partial_{\overrightarrow{n}_x}$, $\nu(\mathbf{X}) = -\Psi(\mathbf{X})$ where ϱ, μ, ν are the densities used in (9.22)–(9.26).

The following is a fundamental identity in the calculus of potentials:

$$\int_S \left[\frac{\partial u(\mathbf{X})}{\partial\overrightarrow{n}_x} E(\mathbf{X}_C, \mathbf{X}) - u(\mathbf{X}) \frac{\partial E(\mathbf{X}_C, \mathbf{X})}{\partial\overrightarrow{n}_x}\right] dS(\mathbf{X}) =$$

$$= q(\mathbf{X}_C)\, u(\mathbf{X}_C), \tag{9.28}$$

where $u(\mathbf{X}_C)$ is harmonic within Ω and belongs to the class $C^1(\Omega \cup \mathrm{S})$, such that $u(\mathbf{X_C})$ is representable in Ω in the form of the sum of Π_{1l} and Π_{2l} where the densities μ and ν are given by $\mu(\mathbf{X}) = \partial u(\mathbf{X})/\partial\overrightarrow{n}_{\mathbf{X}}$, $\nu(\mathbf{X}) = -u(\mathbf{X})$ and fulfill (9.28).

Among BVPs the internal (for Ω^+) and external (for Ω^-) problems posed by Dirichlet and Neumann play a special role. We begin with the BVP of Dirichlet.

- the BVP-DI (DI-Dirichet internal) is of the form

$$\begin{cases} a)\ \Delta u(\mathbf{X}_C) = 0,\ \mathbf{X}_C \in \Omega^+ \cup S,\ u \in C(\Omega^+ \cup S), \\ S \in C^{(1,\alpha)}, \alpha \in (0,1) \\ b)\ u(\mathbf{X}_C) = \varphi^+(\mathbf{X}_C),\ \mathbf{X}_C \in S, \end{cases} \tag{9.29}$$

where φ^{+} is a given function, which is continuous on S. Thus BVP-DI is to find a harmonic solution (function) $\hat{u}$ of (9.29) such that $\hat{u} \in C(\Omega^{+} \cup S)$, $S \in C^{(1,\alpha)}$. The solution $\hat{u}$ of BVP-DI always exists and is unique in the form of Π_{2l}, i.e.

$$\hat{u}_{di}(\mathbf{X}_C) = \int_S \nu(\mathbf{X}) \frac{\partial}{\partial \overrightarrow{n}_X} E(\mathbf{X}_C, \mathbf{X})\, dS(\mathbf{X}), \tag{9.30}$$

Where the density ν is the unique solution of integral second kind Fredholm equation of the form

$$-2^{-1}\nu(\mathbf{X}_C) + \int_S \nu(\mathbf{X}) \frac{\partial}{\partial \overrightarrow{n}_X} E(\mathbf{X}_C, \mathbf{X})\, dS(\mathbf{X}) = \varphi^{+}(\mathbf{X}_C), \tag{9.31}$$

where $\mathbf{X}_C \in S$

- BVP-DE (DE-Dirichlet external) has the form

$$\begin{cases} a)\Delta u(\mathbf{X}_C) = 0,\ \mathbf{X}_C \in \Omega^{-},\ \mathbf{0} \in \Omega^{+},\ u \in C(\Omega^{-} \cup S), \\ S \in C^{(1,\alpha)}, \alpha \in (0,1) \\ b)u(\mathbf{X}_C) = \varphi^{-}(\mathbf{X}_C),\ \mathbf{X}_C \in S, \end{cases} \tag{9.32}$$

φ^{-} given and continous on S, and $\lim_{|\mathbf{X}_C|\uparrow\infty} |\mathbf{X}_C|^{n-2}\, u\,|\mathbf{X}_C| = const.$

The problem of BVP-DE consists of finding a harmonic function $\hat{u}(\mathbf{X}_C)$ in the region Ω^{-}, $\mathbf{0} = \Omega^{+}$, which is regular at infinity, belongs to the class $C(\Omega^{-} \cup S)$, $S \in C^{(1,\alpha)}$, $\alpha \in (0,1)$ and fulfills the conditions given by (9.32).

Under these conditions, the solution $\hat{u}(\mathbf{X}_C)$ exists, is unique and can be represented in the form

$$\hat{u}(\mathbf{X}_C) = \hat{u}_{di}(\mathbf{X}_C) + \beta \cdot |\mathbf{X}_C|^{n-2}, \beta \text{ is a constant}, \tag{9.33}$$

where the density $\hat{u}_{di}(\mathbf{X}_C)$ is the unique solution to the following Fredholm integral equation of the second kind:

$$-2^{-1}\nu(\mathbf{X}_C) + \int_S \nu(\mathbf{X}) \frac{\partial}{\partial \overrightarrow{n}_x} E(\mathbf{X}_C, \mathbf{X})\, dS(\mathbf{X}) = \varphi^{-}(\mathbf{X}_C) - \beta|\mathbf{X}_C|^{2-n}, \tag{9.34}$$

where $\mathbf{X}_C \in S$.

It can be checked that, when $\varphi^{-}(\mathbf{X}_C) = \beta \cdot |\mathbf{X}_C|$, we obtain the solution $\hat{\nu}_0 = 1$, and for (9.34), appropriate choice of β gives

$$\hat{\nu}(\mathbf{X}_{C1}) = \hat{\varphi}^{-}(\mathbf{X}_{C1}) + \beta_1, \beta_1 \text{ any constant, where} \tag{9.35}$$

- $\hat{\nu}^{-}(\mathbf{X}_{C1})$ a denotes particular solution of (9.34),
- $\hat{\beta} = -\int_S \varphi^{-}(\mathbf{X}_C)\,\hat{\nu}_0(\mathbf{X}_C)\,dS(\mathbf{X}_C)$ is the estimate of β,
- the density $\hat{\nu}_0$ fulfills the conditions

  $\int_S \hat{\nu}_0(\mathbf{X}_{C1})\,|\mathbf{X}_{C1}|^{2-n} dS(\mathbf{X}_{C1}) = 1.$

Note 9.12. The density $\hat{\nu}_0$ is also a non-trivial solution of the BVP-NI (to be presented later) with given function $\Psi^{+}(\mathbf{X}_C)$, $\mathbf{X}_C \in S$. For $n \geq 3$, it also fulfills the norming condition of the form

$$\widehat{\Pi}_{1l,0}(\mathbf{X}_C) = \int_S \hat{\nu}_0(\mathbf{X})\,E(\mathbf{X}_C, \mathbf{X})\,dS(\mathbf{X}) = 1,\ \ \mathbf{X}_C \in \Omega^{+} \cup S.$$

The one layer potential $\Pi_{1l,0}$ with density $\nu_0(\mathbf{X_C})$ is often called Robin's potential or equilibrium potential. It should also be added that the density $\hat{\nu}_0(\mathbf{X}_C)$ gives the solution $\hat{u}$ of the so called BVP-R (Robin) or electrostatic problem regarding the distribution of electrical charges on a conductor that produces an equilibrium potential which is constant in the region Ω^{+}. The problem of computational complexity can arise when solving BVP-DE. This follows from the fact that even though u is assumed to be regular at infinity, the harmonic function $u(\mathbf{X}_C)$ can decrease more slowly than $\Pi_{2l}(\mathbf{X}_C)$, i.e. the two-layer potential function. This means, therefore, that $u(\mathbf{X}_C)$ cannot, in general, be represented simply in the form of $\Pi_{2l}(\mathbf{X}_C) = u(\mathbf{X}_C)$.

The second kind of BVPs were formulated by F. Neumann (1887). Let us start with BVP-NI, i.e. the internal Neumann BVP. This can be formalized as follows

- BVP-NI: Find $\hat{u}_{NI}(\mathbf{X}_C)$ which is harmonic in the region Ω^{+}, $\hat{u}_{NI} \in C^1(\Omega^{+} \cup S)$, $S \in C^{(1,\alpha)}$, where $\alpha \in (0,1)$, i.e.

$$\hat{u}_{NI} = \text{solution of} \begin{cases} \triangle u(\mathbf{X}_C) = 0,\ \ \mathbf{X}_C \in \Omega^{+} \\ \frac{\partial u(\mathbf{X}_C)}{\partial \overrightarrow{n}_{X_C}} = \Psi^{+}(\mathbf{X}_C), \mathbf{X}_C \in S, \end{cases} \tag{9.36}$$

where $\Psi^{+}(\mathbf{X}_C)$ is known and continuous on S, i.e. $\Psi^{+} \in C^{\circ}(S)$.

The solution $\hat{u}_{NI}(\mathbf{X}_C)$ exists if and only if the boundary value condition for $\hat{u}_{NI}$ is of the form $\Psi^{+}(\mathbf{X}_C)$ and fulfills the orthogonality condition, i.e. $\int_S \Psi^{+}(\mathbf{X}_C)\,dS(\mathbf{X}_C) = 0$.

This means that

$$\hat{u}_{NI}(\mathbf{X}_C) = \widehat{\Pi}_{1l} + \gamma,\ \text{ where } \gamma \text{ is a constant} \tag{9.37}$$

and the estimate of the one layer potential is:

$$\widehat{\Pi}_{1l}(\mathbf{X}_C) = \int_S \hat{\mu}(\mathbf{X}_C)\,E(\mathbf{X}_C, \mathbf{X})\,dS(\mathbf{X}),$$

where the estimate of the density μ fulfills the following Fredholm integral equation of the second kind:

$$2^{-1}\hat{\mu}\left(\mathbf{X}_C\right)+\int_S \hat{\mu}\left(\mathbf{X}\right)E\left(\mathbf{X}_C,\mathbf{X}\right)dS\left(\mathbf{X}\right)=\Psi^+\left(\mathbf{X}_C\right),\mathbf{X}_C\in S. \qquad (9.38)$$

For $\Psi^+\left(X_C\right)=0$, the solution of (9.38) is $\hat{\mu}_0\left(\mathbf{X}_C\right)$, and $\Psi^+\left(\mathbf{X}_C\right)\neq 0$, and for $\Psi^+\left(X_C\right)\neq 0$, $\Psi^+\left(\mathbf{X}_C\right)\perp dS\left(\mathbf{X}_C\right)$, the general solution of (9.38) is $\hat{\mu}\left(\mathbf{X}_C\right)+\gamma_0\hat{\mu}_0\left(\mathbf{X}_C\right)$, where γ_0 is any constant.

Note 9.13. There are many formulas for the solutions of each of BVP-DI, BVP-DE, BVP-NI and BVP-NE, since we may consider various analytical forms of $\varphi^+,\varphi^-,\Psi^+$ and Ψ^- that are assumed to be known and empirically observable, as well as various definitions of S, $E\left(\mathbf{X}_C,\mathbf{X}\right)$ and u. In order to reflect reality well, the choice of all of these formal objects is important, especially the choice of u.

The second Neumann problem which is often analyzed is

- BVP-NE: Find a harmonic function $\hat{u}_{NE}\left(X_C\right)$, which is harmonic in the region $\Omega^-, 0\in\Omega^+$, $\hat{u}_{NE}\in C^{-1}\left(\Omega^-\cup S\right)$, $S\in C^{(1,\alpha)}$, $\alpha\in(0,1)$, such that $\hat{u}_{NE}$ satisfies

$$\hat{u}_{NE}\left(\mathbf{X}_C\right)\equiv \text{ solution of } BVP\begin{cases}\triangle u\left(\mathbf{X}_C\right)=0,\mathbf{X}_C\in\Omega^-,\\ \frac{\partial u(\mathbf{X}_C)}{\partial\overrightarrow{n}_{XC}}=\Psi^-\left(\mathbf{X}_C\right),\mathbf{X}_C\in S,\end{cases} \qquad (9.39)$$

where $\Psi^-\in C^{\circ}\left(S\right)$ is a known function and, like u, is also regular at infinity.

It is worth recalling that for $n=2$ the solution $\hat{u}_{NE}\left(\mathbf{X}_C\right)$ (given with accuracy up to an additive constant) exists if and only if

$$\int_S \Psi^-\left(\mathbf{X}_C\right)dS\left(\mathbf{X}_C\right)=0. \qquad (9.40)$$

For $n\geq 3$, the solution $\hat{u}_{NE}\left(\mathbf{X}_C\right)$ always exists and is unique.

Moreover, the solution $\hat{u}_{NE}\left(\mathbf{X}_C\right)$ is representable in the form of a single layer potential, i.e.

$$\hat{u}_{NE}\left(\mathbf{X}_C\right)=\int_S \hat{\mu}\left(\mathbf{X}\right)E\left(\mathbf{X}_C,\mathbf{X}\right)dS\left(\mathbf{X}\right)$$

with $\hat{\mu}$ being the solution of the following Fredholm integral equation of the second kind

$$-2^{-1}\mu\left(\mathbf{X}_C\right)+\int_S \mu\left(\mathbf{X}\right)\frac{\partial}{\partial n_{X_C}}E\left(\mathbf{X}_C,\mathbf{X}\right)dS\left(\mathbf{X}\right)=\Psi^-\left(\mathbf{X}_C\right),\mathbf{X}_C\in S. \qquad (9.41)$$

The solution $\hat{\mu}$ of (9.41) always exists and is unique for $n \geq 3$.

For $n = 2$ and $\Psi^-(\mathbf{X}_C) = 0$, $\hat{\mu}_0(\mathbf{X}_C)$ is a nontrivial solution. Thus, if $\Psi^-(\mathbf{X}_C) \neq 0$ satisfies(9.40), the unique solution $\hat{\mu}(\mathbf{X}_C)$ of (9.41) fulfills the condition $\int_S \hat{\mu}(\mathbf{X}_C) dS(\mathbf{X}_C) = 0$, and the general solution of (9.41) is of the form $\hat{\mu}_g(\mathbf{X}_C) = \hat{\mu}(\mathbf{X}_C) + \gamma_g \hat{\mu}_0(\mathbf{X}_C)$, where γ_g is any constant.

Note 9.14. G. Bouligand (1926) presented a brief and clear analysis of the relations between harmonic functions and BVP, according to the principles of Picard and Dirichlet. This paper gives a broad discussion of extensions of these principles by the following: Bouligand (13 papers), O. D. Kellog (6 papers), H. Lebesgue (8 papers), E. Picard (6 papers), S. Zaremba (13 papers), N. Wiener (4 papers) among many others.

According to Bouligand, it was Zaremba who made "le premier pas décisif dans l'extension du principe de Dirichlet".

Especially interesting are Bouligand's notes on the close connections of many concepts from mathematical analysis, functional analysis and topology with such concepts as Green's functions, discontinuous boundary condition functions and potential functions. It should be stressed that solving BVPs may be achieved using Green's (1828) functions. In the case of BVP-DI, such a function takes the form

$$G(\mathbf{X}_C, \mathbf{X}) = E(\mathbf{X}_C, \mathbf{X}) + h(\mathbf{X}_C, \mathbf{X}), \mathbf{X}_C \in \Omega^+ \cup S, \mathbf{X} \in \Omega^+,$$

where $h \in C^\circ(\Omega^+ \cup S)$ in $\mathbf{X}_C$ and is harmonic in Ω^+. For $\mathbf{X} \in \Omega^+$, $G(\mathbf{X}_C, \mathbf{X}) = 0$ if $\mathbf{X}_C \in S$. Therefore, for the Poisson equation of the form $\triangle u = f(\mathbf{X}_C)$, $\mathbf{X}_C \in \Omega^+$ with boundary condtion $u(\mathbf{X}_C) = \varphi^+(\mathbf{X}_C)$, $\mathbf{X}_C \in S$, $\hat{u}_{di} \in C^2(\Omega^+) \cap C(\Omega^+ \cup S)$ can be represented by:

$$\hat{u}_{di}(\mathbf{X}_C) = \int_{\Omega^+} f(\mathbf{X}) G(\mathbf{X}_C, \mathbf{X}) d\mathbf{X} + \int_S \varphi^+(\mathbf{X}) \frac{\partial}{\partial n_x} G(\mathbf{X}_C, \mathbf{X}) dS(\mathbf{X}),$$

where $\mathbf{X}_C \in \Omega^+$ and for further calculus of potentials, it is useful to compute the values of the following integrals with $\mathbf{X}_C$ as their parameter:

$$\Pi^d_{Gvol}(\mathbf{X}_C) = \int_\Omega \varrho(\mathbf{X}) G(\mathbf{X}_C, \mathbf{X}) d\mathbf{X}, \mathrm{G}_{2l}(\mathbf{X}_C) =$$

$$= \int_S \nu(\mathbf{X}) \frac{\partial}{\partial n_x} G(\mathbf{X}_C, \mathbf{X}) dS(\mathbf{X}).$$

9.4. Non-classical results

Extensions and generalizations of concepts in the field of potential theory were made possible due to the development of measure theory and concepts of integrals, especially Radon integrals.

Note 9.15. For the positive Borelian measure $\lambda_B \geq 0$ acting on R_n with compact support supp λ, the potential of the measure λ_B, $\Pi_{\lambda_B}(\mathbf{X}_C)$, can be defined as

$$\Pi_\lambda(\mathbf{X}_C) \equiv \Pi_{\lambda_B}(\mathbf{X}_C) =$$

$$= \int_{\mathrm{R}^n} E(\mathbf{X}_C, \mathbf{X}) d\lambda_B(\mathbf{X}), \begin{cases} \Pi_\lambda : R_n \to [0, \infty] \text{ if } n \geq 3 \\ \Pi_\lambda : R_2 \to (-\infty, \infty] \text{ if } n = 2. \end{cases} \tag{9.42}$$

The potential Π_λ exists everywhere in R^n and is a superharmonic function in R^n, and harmonic on the exterior of supp(λ). For λ of any sign and compact support supp$\Pi_\lambda(-\mathbf{X}_C)$, using the canonical decomposition of the measure λ, $\lambda = \lambda^+ - \lambda^-$, where $\lambda^+ \geq 0$ and $\lambda^- \geq 0$, we obtain $\Pi_\lambda = \Pi_{\lambda^+} - \Pi_{\lambda^-}$. The potential Π_λ is not determined for $\Pi_{\lambda^+} = \Pi_{\lambda^-} = \infty$. If $\lambda \geq 0$ is centered on a smooth surface S, then $\Pi_{2l,\lambda}$ fulfills the equation $\frac{\partial \Pi_\lambda(\mathbf{X}_C)}{\partial n_X} = \int \frac{\partial}{\partial n_X} E(\mathbf{X}_C, \mathbf{X})\, d\lambda(\mathbf{X})$. It was found that in R_n, $\Pi_\lambda(\mathbf{X}_C) < \infty$ except at points of a polar set of measure zero.

Note 9.16. For a general function or distribution, i.e. a function G in R_n, the potential of G is defined as the convolution $E * G$, where E is a convolution function (kernel).

More detailed results can be found, e.g in W. S. Wladimirow (1979, pp. 83, 142, 219–223). The following are some of them:

Example 3 Let H_s^2 be a Hilbert space of functions $h(\xi)$ with

$$\|h\|_{(s)} = \left[\int |h(\xi)|^2 \cdot \left(1 + |\xi|^2\right)^s d\xi\right]^{1/2} = \left\| g(\xi) \cdot \left(1 + |\xi|^2\right)^{s/2} \right\|$$

and $F_{(s)} \equiv \left\{ f(x) \,|\, f(x) = Fh(\xi), h \in H_s^2 \right\}$, $\|f\|_{(s)} = \|h\|_{(s)}$, $s \in R$.

For the kernel $K_s(X) = F^{-1}\left[\left(1 + |\xi|^2\right)^{-s/2}\right]$, the convolution defining the Bessel potential function is: $\Pi_{Bess}(x_c) \equiv f(x_c) = h(\xi) * K_s(x_c)$, $h \in H^2$, and the mapping $h \to f = h * K_s$ is reciprocally unique and continuous from H^2 onto F_s.

Example 4 Let $f \in F$, $\mathbf{X}_C \in \Omega_0$, $n = 2$. Then $\Pi(\mathbf{X}_C) = \left(ln|\mathbf{X}_C|^{-1}\right) * f$ is a logarithmic potential with generalized density f. If $n \geq 3$, then

$\Pi_n(\mathbf{X}_C) = |\mathbf{X}_C|^{2-n} * f$ is called the generalized Newtonian potential. For $f = \varrho(\mathbf{X}_C)$, where ϱ is a summable, finite function, $\mathbf{X}_C \in R_n$, $n \geq 3$, then Π_n is a simple volume potential, and $\Pi_n(\mathbf{X}_C) = \int \varrho(\mathbf{X}) \cdot |\mathbf{X}_C - \mathbf{X}|^{2-n} d\mathbf{X}$.

In the case when $f \in F^1_{loc}\left(\overline{R}_{+,1} \star \times R\right)$, the convolution $f * E_n(\mathbf{X}_C, \mathbf{X}) = \Pi_n(\mathbf{X}_C)$ is called the wave potential with generalized density f, and

$$E_n(\mathbf{X}_C, \mathbf{X}) = \left[2^{2n-1}\Pi^n\Gamma(n)\right]^{-1} * {}^{n-1}\left[\theta(\mathbf{x}_0)\,\varrho\left(\mathbf{x}_C^2\right)\right], \overline{n} = 2n+1, n \geq 1$$

$$E_n(\mathbf{X}_C, \mathbf{X}) \equiv \left[2^{2n-1}\Pi^n\Gamma(n)\right]^{-1} * {}^{n-1}\left[\theta(\mathbf{x}_0)\,\mathbf{x}_C^2\right]^{-\frac{1}{2}}, \overline{n} = 2n$$

$$E_n(\mathbf{X}_C, \mathbf{X}) = 2^{-1}\theta(\mathbf{x}_0)\,\theta\left(\mathbf{x}_c^2\right), n = 1.$$

The wave potentials of surfaces can also be defined for Cauchy problems of heat conduction where convolution with generalized functions is used extensively. For readers with strong backgrounds in topology and functional analysis, the following texts are recommended: N.S. Landkoff (1966), J. Wermer (1974), L. L. Helms (2009), M. Brelot (1967, 1971), H. Cartan (1945, 1946) and A. Björn, J. Björn (2014).

There are also particularly important further generalizations of the results of S. Zaremba, H. Lebesgue and M. Brelot for cases where the appropriate measures are sub or super harmonic, but the boundaries have cusps, or when generalized Green's functions are computationally applicable.

9.5. Measuring potential in economics

The category of potential is not so broadly used in economics as in physics. In the last 40 years, however, due to great progress in the development of theoretical and applied econometrics, as well as mathematical economics, this concept is being applied more and more often. There is strong pressure from practically oriented economic and financial leaders, who wish to describe the potential of the economy, sectors or branches of the economy, markets for consumption or investment goods. In response to this, there have been many efforts by econometricians, economists and financiers to measure such potential.

It is worthwhile to underline that current methods for measuring potential for a concrete object in the field of economics normally use tools from the time-domain analysis of time series, i.e. based on autocorrelation and cross-correlation. The Hodrick-Prescott filter is very popular in this respect. Their idea does not use a model of the mutual connections between economic variables and is based on removing cyclical fluctuations from data to assess the underlying long-term trend, i.e. potential.

The presentation given below is an attempt to formulate another approach, which is model-based. Here, we will use the following notation:

$Y(t) \equiv GDP(t)$ – Gross Domestic Product in the periods $t = t_0, ..., t_a$, where t_0 is the first period studied, t_a is the current period or the last period studied,

$X_1(t) \equiv K_{ph}(t)$ – physical capital in the period t,

$X_2(t) \equiv K_h(t)$ – human capital in the period t,

$a, b, c, a_0, a_1, a_2 \in R$ – parameters whose values belong to the set of real numbers,

$y(t), x_1(t), x_2(t) \in R$ – the observed values of variables Y, X_1, X_2 in the period t.

We will consider two versions of a data generating process for $\{Y(t)\}_{t=t}^{t_a}$, each based on one of the following production functions:

$pf1)$ $Y(t) = a_0 X_1^{a_1}(t) X_2^{a_2}(t)$,
$pf2)$ $Y(t) = aX_1^2(t) + bX_1(t)X_2(t) + cX_2^2(t)$.

In the case of $(pf1)$, using the tools of mathematical analysis, we obtain formal expressions with a suitable economic meaning. Thus from the necessary conditions for the existence of $max\{Y(t)\}$ and the condition for the equality of the first partial derivatives of $Y(t)$ with respect to $X_1(t)$ and $X_2(t)$, as well as the conditions that $a_0 \neq 0$; $a_1 \neq 0; 1$ and $a_2 \neq 0; 1$, we obtain the following formula:

$$\tilde{X}_{1,1}(t) = a_1 a_2^{-1} X_2(t). \tag{9.43}$$

This formula enables us to calculate this part of the theoretical potential value of GDP which guarantees the equality of very small relative changes in $Y(t)$ with respect to very small changes in $X_1(t)$, symbolizing physical capital expressed in monetary values, and with respect to very small changes in $X_2(t)$, i.e. human capital. Thus, the considered partial potential value of $GDP \equiv Y$ is calculated according to the formula

$$\tilde{Y}_{1,1}(t) = a_0 \tilde{X}_1^{a_1}(t) X_2^{a_2}(t). \tag{9.44}$$

By embedding $(pf1)$ into a model with a stochastic structure, we obtain

$$Y(t) = a_0 X_1^{a_1}(t) X_2^{a_2}(t) exp(\Xi(t)), \Xi(t) \sim N(0, \sigma^2), \tag{9.45}$$

where the random specification error $\Xi(t)$ has a normal distribution with expected value 0 and variance σ^2. Using the method of Gauss-Legendre estimation, we can find estimates $\hat{a}_0, \hat{a}_1, \hat{a}_2$ of a_0, a_1, a_2 based on the sample.

Using these estimates, we may write the empirical version of eq. (9.44) as

$$\tilde{Y}_{1,1}(t) = \hat{a}_0 \hat{\tilde{X}}_{1,1}^{\hat{a}_1}(t) X_2^{\hat{a}_2}(t), \tag{9.46}$$

where $\hat{\tilde{X}}_{1,1}(t) = \hat{a}_1 \hat{a}_2^{-1} X_2(t)$.

The second component of potential value of GDP can be derived from the Laplace condition (see Definition 1 in this chapter)

$$\frac{\delta^2 Y(t)}{\delta X_1(t)^2} = \frac{\delta^2 Y(t)}{\delta X_2(t)^2} \tag{9.47}$$

and the conditions $a_0 \neq 0$; $a_1 \neq 0; 1; 2$ and $a_2 \neq 0; 1; 2$, i.e. from the sufficient conditions for the existence of an extreme point.

This condition requires the equality of marginal changes in GDP with respect to physical capital and with respect to human capital, both expressed in monetary values.

The counterparts of (9.43), (9.44) and (9.46) for the case (9.45) are as follows:

$$\tilde{X}_{1,2}(t) = \left[\frac{a_1(a_1 - 1)}{a_2(a_2 - 1)}\right]^{\frac{1}{2}} X_2(t), \tag{9.48}$$

$$\tilde{Y}_{1,2}(t) = a_0 \tilde{X}_{1,2}^{a_1}(t) X_2^{a_2}(t), \tag{9.49}$$

and the empirical estimates are of the form

$$\hat{\tilde{X}}_{1,2}(t) = \left[\frac{\hat{a}_1(\hat{a}_1 - 1)}{\hat{a}_2(\hat{a}_2 - 1)}\right]^{\frac{1}{2}} X_2(t), \tag{9.50}$$

$$\hat{\tilde{Y}}_{1,2}(t) = \hat{a}_0 \hat{\tilde{X}}_{1,2}^{\hat{a}_1}(t) X_2^{\hat{a}_2}(t). \tag{9.51}$$

Formulas (9.48) – (9.51) will change if we directly use Laplace's equation $\Delta Y(t) = 0$, i.e. if we use

$$\frac{\delta^2 Y(t)}{\delta X_1(t)^2} = \frac{\delta^2 Y(t)}{\delta X_2(t)^2}. \tag{9.52}$$

In this case, we obtain the following formulas:

$$\tilde{X}_{1,2a}(t) = \left[\frac{a_1(a_1-1)}{a_2(1-a_2)}\right]^{\frac{1}{2}} X_2(t), \tag{9.53}$$

$$\tilde{Y}_{1,2a}(t) = a_0 \tilde{X}_{1,2}^{a_1}(t) X_2^{a_2}(t), \tag{9.54}$$

$$\hat{\tilde{X}}_{1,2a}(t) = \left[\frac{\hat{a}_1(\hat{a}_1-1)}{\hat{a}_2(\hat{a}_2-1)}\right]^{\frac{1}{2}} X_2(t), \tag{9.55}$$

$$\hat{\tilde{Y}}_{1,2a}(t) = \hat{a}_0 \hat{\tilde{X}}_{1,2a}^{\hat{a}_1}(t) X_2^{\hat{a}_2}(t). \tag{9.56}$$

It should be stressed that $\tilde{Y}_{1,2a}(t)$ and $\hat{\tilde{Y}}_{1,2a}(t)$ fulfill Laplace's equation outside of the limited area of the region of the values of both capitals where the interaction between the two types of capital is productive. This signifies that the empirical estimate $\hat{\tilde{Y}}_{1,2a}(t)$ may be treated as an estimate of this part of potential value of GDP which can be attributed to the potential simultaneous influence of both forms of capital.

Note 9.17. In order to guarantee the positivity of the values of $\tilde{Y}_{1,2a}(t)$ and $\hat{\tilde{Y}}_{1,2a}(t)$, the following conditions also need to be satisfied:

- when $a_1, a_2; \hat{a_1}, \hat{a_2} > 0$, then either $a_1 > 1; 1 > a_2$ or $\hat{a}_1 > 1; 1 > \hat{a}_2$,
- when $a_1, a_2 < 0; \hat{a_1}, \hat{a_2} < 0$, then either $a_1 < 1; 1 < a_2$ or $\hat{a}_1 < 1; 1 < \hat{a}_2$,
- when $a_1 > 0, a_2 < 0; \hat{a_1} > 0, \hat{a_2} < 0$, then either $a_1 < 1; 1 < a_2$ or $\hat{a}_1 < 1;\ 1 < \hat{a}_2$,
- when $a_1 < 0, a_2 > 0; \hat{a_1} < 0, \hat{a_2} > 0$, then either $a_1 > 1; 1 < a_2$ or $\hat{a}_1 > 1;\ 1 < \hat{a}_2$.

Note 9.18. The function $Y(t)$ from $(pf1)$ has a strict minimum at $(0,0)$ under the above assumptions, but it does not have a strict maximum. This means that for $(pf1)$ we can not find a potential function in the sense the maximum possible level, i.e. to find the maximal potential values of GDP for each pair of positive values of $X_1(t), X_2(t), t = t_0, ..., t_a$, together with the estimates $\hat{a}_0, \hat{a}_1, \hat{a}_2$.

For the production function $(pf2)$, from the necessary conditions for an extreme point, we obtain

$$\tilde{X}_{1,1}(t) = \frac{2c-b}{2a-b} X_2(t) \tag{9.57}$$

$$\hat{\tilde{X}}_{1,1}(t) = \frac{2\hat{c} - \hat{b}}{2\hat{a} - \hat{b}} X_2(t), \tag{9.58}$$

where $\hat{a}, \hat{b}$ and $\hat{c}$ are the least-squares estimates of the parameters a, b and c, respectively (as defined by $(pf2)$),

$$\tilde{Y}_{1,1}(t) = a\tilde{X}_{1,1}^2(t) + b\tilde{X}_{1,1}(t)X_2(t) + cX_2^2(t) \tag{9.59}$$

and the empirical potential GDP function is given by:

$$\hat{\tilde{Y}}_{1,1}(t) = \hat{a}\hat{\tilde{X}}_{1,1}^2(t) + \hat{b}\hat{\tilde{X}}_{1,1}(t)X_2(t) + \hat{c}X_2^2(t). \tag{9.60}$$

The function $(pf2)$ has a maximum if $a < 0$ and $a > \frac{b^2}{4c}$, i.e. the individual influence of the square of the value of physical capital $\tilde{X}_1$ reduces the estimated value of the potential GDP. These conclusions follow from the sufficient condition that $det \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} = 4ac - b^2 > 0$, $4ac > b^2$, and $\frac{\delta^2 Y}{\delta X_1^2} < 0$, i.e. $a < 0$.

Note 9.19. In the case where the production function takes the form

$$Y(t) = aX_1^2(t) + bX_1(t)X_2(t) + cX_2^2(t), \tag{9.61}$$

and we use both the necessary and sufficient conditions for the existence of a maximum, we obtain

$$\tilde{X}_{1,1}(t) = 2acb^{-2}, \tag{9.62}$$

$\tilde{Y}_{1,1}(t) = a\tilde{X}_{1,1}^2(t) + b\tilde{X}_{1,1}(t)X_2(t) + cX_2^2(t)$ and their empirical versions

$$\hat{\tilde{X}}_{1,1}(t) = 2\hat{a}\hat{c}\hat{b}^{-2}, \tag{9.63}$$

$\hat{\tilde{Y}}_{1,1}(t) = \hat{a}\hat{\tilde{X}}_{1,1}^2(t) + \hat{b}\hat{\tilde{X}}_{1,1}(t)X_2(t) + \hat{c}X_2^2(t)$, where, in practice, $\hat{\tilde{Y}}_{1,1}(t)$ denotes the empirical potential part of GDP computed according to (9.60), $\hat{a}, \hat{b}, \hat{c}$ denote the least-squares estimators of the parameters a, b, c of the production function.

It is open question as to what degree the functions $pf1, pf2$ can serve as a basis for calculating the components of estimates of potential for Poland or other countries. Initial empirical trials show that they provide a reasonable basis for calculating suitable shares in production, $\hat{\tilde{Y}}_{1,1}(t)/Y(t)$ and

$\hat{\hat{Y}}_{1,2}(t)/Y(t)$, and shares in growth, $gr(\hat{\hat{Y}}_{1,1}(t))/gr(Y(t))$ and $\hat{\hat{Y}}_{1,2}(t)/Y(t)$ for Poland, the EU, GB and USA.

From empirical point of view, it is important to check the practical and theoretical validness of our methods for computing the components of the production potential for a given economy and time. This approach uses the following definitions of admissible transformations of the ratio $\hat{\chi}_{12}^{(1)}(t) \equiv \frac{X_1(t)}{X_2(t)}$, interpreted here as the endowment of one worker with a suitable monetary value of physical capital. We introduce the following modifications of $\hat{\chi}_{12}^{(1)}(t)$:

$$\hat{\chi}_{12}^{(2)}(t) \equiv [1 - R^{cr}(t)]\, \hat{\chi}_{12}^{(1)}(t),$$

$$\hat{\chi}_{12}^{(3)}(t) \equiv \left[1 - R^{cr}(t) - 2^{-1}(\overline{CIT}(t) + \overline{PIT}(t))\right] \hat{\chi}_{12}^{(1)}(t),$$

$$\hat{\chi}_{12}^{(4)}(t) \equiv \dot{\chi}_{12}(t)\hat{\chi}_{12}^{(1)}(t), \dot{\chi}_{12}(t) \equiv \frac{\dot{X}_1(t)}{\dot{X}_2(t)}, \dot{X}_i(t) \equiv \Delta X_i(t)$$

$$\hat{\chi}_{12}^{(5)}(t) \equiv \dot{\chi}_{12}(t)\hat{\chi}_{12}^{(1)}(t), \dot{\chi}_{12}(t) \equiv \frac{\dot{I}_1(t)}{\dot{I}_2(t)}, \frac{\dot{TD}_\varphi(t)}{\dot{TD}_h(t)}$$

$$\hat{\chi}_{12}^{(6)}(t) \equiv \ddot{\chi}_{12}(t)\hat{\chi}_{12}^{(1)}(t), \ddot{\chi}_{12}(t) \equiv \frac{\ddot{X}_1(t)}{\ddot{X}_2(t)}, \ddot{X}_i(t) \equiv \Delta^2 X_i(t)$$

$$\hat{\chi}_{12}^{(7)}(t) \equiv (1 - \gamma_{GINI}(t))\hat{\chi}_{12}^{(1)}(t),$$

where:

R^{cr} is the interest rate for credit (for example, LIBOR, FEDIR, FIBOR or WIBOR),

$\overline{CIT}; \overline{PIT}$ – the mean rates of corporate and personal income, respectively; tax for given economy,

$\dot{I}_i(t) \equiv \Delta I_i(t)$, where I is the monetary value of investments in K_{ph} or K_h (appropriate to the index i),

$\dot{TD}_i(t) \equiv \Delta TD_i(t)$, TD_i total debt in φ – firms or h – households,

γ_{GINI} – GINI index of inequality expressed in decimal numbers.

Note 9.20. It is possible to consider the monetary value of actively used capital: $X_3(t) \equiv K_f(t)$ financial capital or $\underline{K}_f(t)$ unemployed financial capital, as well as

$X_4(t) \equiv K_h(t)$ or $\underline{K}_h(t)$ active or non-active human capital,

$X_5(t) \equiv K_s(t)$ or $\underline{K}_s(t)$ active or non-active social capital.

The second group of elements concerns the analytical forms of the parts of production potential, e.g. the forms

$$\hat{\hat{Y}}_{2,a}^{(j)} = \hat{a}_0 X_1^{\hat{a}_1}(t) X_2^{\hat{a}_2}(t) + \hat{d}_0 \underline{X}_{1,j}^{\hat{d}_1}(t) \underline{X}_2^{\hat{d}_2}(t), j = 1, 2, ..., 7,$$

where the non-active physical capital is definied as

$$\underline{X}_1^{(1)}(t) = \hat{\chi}_{12}^{(1)}(t) \underline{X}_2(t)$$

or

$$\underline{X}_1^{(j)}(t) = \hat{\chi}_{12}^{(j)}(t) \underline{X}_2(t), j = 2, 3, \ldots, 7,$$

where $\underline{X}_2(t)$ – number of unemployed persons of working age, $\hat{a_0}, \hat{a_1}, \hat{a_2}, \hat{d}_0, \hat{d}_1, \hat{d}_2$ are estimates of $a_0, a_1, a_2, d_0, d_1, d_2$ – the parameters of the Wicksell-Cobb-Douglas production function based on the observed values of $X_1(t), X_2(t)$, $\underline{X}_1(t)$, $\underline{X}_2(t)$ and $t = t_1, ..., t_n$ are time periods.

Setting $\hat{\hat{Y}}_{2,a}^{(j),*} = \hat{d}_0 \underline{X}_{1,j}^{\hat{d}_1}(t) \underline{X}_2^{\hat{d}_2}(t)$, we may define the share of unrealized part of production potential as the value of the following ratio:

$$\hat{\sigma}^{\pi,j}(t) = \frac{\hat{\hat{Y}}_{2,a}^{(j),*}}{\hat{\hat{Y}}_{2,j}^{(j)}}.$$

Note 9.21. These propositions may be extended to include other forms of capital (e.g. $X_3(t), X_4(t), X_5(t)$), and other forms of production functions, such as:

$$\hat{\hat{Y}}(t) = \hat{a_0} X_1^{\hat{a_1}}(t) X_2^{\hat{a_2}}(t) + \hat{b_0} X_{1,EU}^{\hat{b_1}}(t) \underline{X}_2^{\hat{b_2}}(t) +$$

$$+ \hat{c_0} \underline{X}_{1,EU}^{\hat{c_1}}(t) \underline{X}_2^{\hat{c_2}}(t) + \hat{d}_0 \underline{X}_1^{\hat{d}_1}(t) \underline{X}_2^{\hat{d}_2}(t),$$

where $X_{1,EU}(t)$ – is the observed monetary value of physical capital in the EU (European Union) at time t.

9.5.1. Examples of empirical applications

Note 9.22. The first empirical applications of $\hat{\chi}_{12}^{(1)}(t)$ to data for the Polish and UK, Germany and European Union economies are promising in terms of the similar patterns of dynamic changes in the potential parts of GDP real quantities for these countries.

In empirical applications we consider two versions of data generating process for production $pf1$ and $pf2$. We distinguish the following methods $M1$, $M2$, $M3$ of calculating partial potential of GDP:

$M1$ – based on the $pf1$ type production function we obtain two versions of potential GDP:

$v1$ – using the estimates of coefficients of (9.43), (9.46) we obtain the empirical version of partial potential according to formula (9.46),

$v2$ – second version is obtained from estimates of (9.50) and (9.51),

$M2$ – using the production function of type $pf2$ the empirical versions of potential are calculated from (9.58) and (9.60),

$M3$ – we examine some modifications of the formula for $\hat{\chi}_{12}^{(1)}(t)$ of type $\hat{\chi}_{12}^{(2)}(t) \equiv [1 - R^{cr}(t)]\, \hat{\chi}_{12}^{(1)}(t)$.

All estimates were made on the yearly data covering period 1995–2015.

In the table 9.1 and 9.2 we present the results of potential calculus estimated for European Union, Germany, Poland, United Kingdom for 2015.

Table 9.1. Empirical results of GDP potential growth rates (in %)

Country	GDP growth rate in 2015	GDP partial potential growth rates			
		$M1\ v1$	$M1\ v2$	$M2$	$M3$
European Union	2,31	0,16	0,16	0,32	1,01
Germany	1,74	0,31	0,31	0,62	0,37
Poland	3,84	-0,24	-0,24	-0,48	1,97
United Kingdom	2,35	0,66	0,66	1,33	1,67

Source: own calculations.

Table 9.2. Empirical results of the share of partial potential production in the real GDP (in %)

Country	GDP partial potential growth rates			
	$M1\ v1$	$M1\ v2$	$M2$	$M3$
European Union	4,50	5,91	56,81	9,51
Germany	3,96	4,41	27,12	12,29
Poland	4,59	2,40	8,18	11,53
United Kingdom	5,78	4,83	47,50	12,60

Source: own calculations.

In the calculations following approximations of variables were used:

- R^{cr} is one month interest rate for credit for the European Union, Germany, Poland and the United Kingdom respectively,

- $Y(t)$ is gross domestic product (chain link volumens 2010=100, millions of national currency) for the European Union, Germany, Poland and the United Kingdom respectively,
- $X_1(t)$ is gross capital formation (chain link volumens 2010=100, millions of national currency) for the European Union, Germany, Poland and the United Kingdom respectively,
- $X_2(t)$ is active population (thousands of persons) for the European Union, Germany, Poland and the United Kingdom respectively.

Economic conclusions based on the above described results of calculations are as follows:

1) obtained estimates of partial potential GDP calculus can be interpreted as the values theoretically possible to realize and which may be obtained additionally above the empirically observed;
2) these estimations according to the methods $M1$ and $M3$ are resulting from the constancy of velocity changes, and for method $M2$ from the condition of acceleration constancy; For the $M3$ additionally they are caused by changes in central bank interest rates;
3) dynamics of the partial potential GDP is best estimated by the $M3$;
4) the $M2$ estimates indicate the greater degree of unused capacity of production forces than other methods (for Poland c.a. 15%, GB over 50%, EU almost 50%, Germany almost 10%). Only in the case of Germany economy the $M3$ shows the greater unused capacity of production (almost 14%) than the $M2$ (almost 10%);
5) the estimates obtained according to the $M1$ (both versions) indicate that, for each economy, the unused capacity of production is the least, and that the share of theoretically possible to obtain additional partial potential production does not exceed 10% of real GDP.

Bibliography

Aharonov, Y., Bohm, D. (1959). Significance of electromagnetic potentials in quantum theory. *Phys. Rev.*, T. 115, pp. 485–491.

Alembert, J. de R. (1743). *Traité de dynamique.* Paris.

Anger, G. (1957). *Stetige Potentiale und deren Verwendung für einen Neuaufbau der Potentialtheorie.* Thesis Dresden.

Appell, P. (1984). Sur les fonctions de trois variables réelles satisfaisant a' l'équation $\triangle F = 0$. *Acta Mathematica*, T. 4, pp. 313–374.

Arsove, M. G. (1953). Functions of potential type. *Trans. Amer. Math. Soc.* T. 75, pp. 526–551.

Arzela, C. (1897). *Sul principio di Dirichlet.* Rendic. Acc. Bologna.

Bauer, H. (1958). Un probléme dr Dirichlet pour la frontiére de Šilov d'un espace compact. *C. R. Acad. Sci.* T. 247, pp. 843–846.

Bauer, H. (1966). *Harmonische Räume und Potentialtheorie.* Springer Berlin.

Björn, A., Björn, J. (2014). The variational capacity with respect to nonopen sets in metric space. *Potential Analysis*, T 40, pp. 57–80.

Bocher, M. (1903). Singular points of functions which satisfy partial differential equations of the elliptic type. *Bull. Amer. Math. Soc.*, T. 9, pp. 455–465.

Bouligand, G. (1926). Fonctions harmoniques. Principes de Picard et de Dirichlet. Mémorial des sciences mathematiques. *Fascicule* 11, pp. 1–52, Gauthier-Villars

Bouligand, G. (1924). Sur les principes de la théorie du potentiel. *Bull. Science Math.*, T 48, Juillet.

Bouhalov, I. (2009). *Phisica inertyi i gravitazyi.* URSS, Moskva.

Brelot, M. (1938). Sur le potentiel et les suites de fonctions sousharmonique. *C. R. Acad. Sci.*, T. 207, p. 836.

Brelot, M. (1960). *Lecture on potential theory.* Tata Inst. Bombay.

Bunge, M. (2014). *Does the Aharonov-Bohm effect occur?* Found. Sci.

Bunge, M. (1971). *On topologies and boundaries in potential theory.* Springer Berlin.

Calderon, A. (1950). On the behavior of harmonic functions at the boundary. *Trans. Amer. Math. Soc.*, T. 68, pp. 47–54.

Carleson, L. (1958). On the connection between Hausdorff measures and capacity. *Ark. F. Math.*, T. 3, pp. 403–406.

Cartan, H. (1941). Sur les fondaments de la théorie du potentiel. *Bull. Soc. Math.*, France, T. 69, pp. 71–96.

Cartan, H. (1945). Théorie du potentiel newtonien. *Bull. Soc. Math. France*, T. 73, pp. 74–106.

Cartan, H. (1946). Théorie générale du balayage en potentiel newtonien. *Ann. Univ. Grenobble*, T. 22, pp. 221–280.

Choquet, G. (1955). Theory of capacities. *Ann. Inst. Fourier*, T. 5, pp. 131–295.

Choquet, G. (1957). Sur les fondaments de la theorie fine du potentiel. *C. R. Acad. Scie.*, T. 244, pp. 1606–1609.

Choquet, G. (1958). Capacibilité en potentiel logarithimique. *Bull classe Sci. Bruxelles*, T. 44, pp. 321–326.

Choquet, G., Deny, J. (1956/1957). Modéles finis en theorie du potential. *J. Anal. Math.*, T. 5, pp. 77–135.

Constantinescu, C., Cornea, A. (1972). *Potential theory on harmonic spaces.* Springer, Berlin.

Deny, J. (1950). Les potentiels d' énergie finie. *Acta Math.*, T. 82, pp. 107–183.

Doob, J. L. (1962) Boundary properties of functions with finite Dirichlet integral. *Ann. Inst. Fourier*, T. 12, pp. 573–621.

Doob, J. L. (1984). *Classical potential theory and its probabilistic counterpart.* Springer N.Y.

Evans, G. C. (1921). Problems of potential theory. *Proc. Amer. Acad. Sc.*, T. 7, pp. 89–98.

Fredholm, I. (1900). *Sur une nouvelle méthode pour la resolution du problčme de Dirichlet.* Ofr. Kgl. Vet. Ak. Förh., Stockholm.

Gauss, C. F. (1840) *Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden.*

Gevrey, M. (1920). Sur la determination des fonctions dr Green. *C. R. Ac Sc.*, T. 171, October, p. 609.

Gréen, G. (1828). *An essay on the application of mathematical analysis to the theories of electricity and magnetism.* Nottingham, Anziehungs- und Abstossungs-Kräfte, In Werke (pp. 195–242). Springer Berlin Heidelberg.

Hailperin, T (2001). Potential infinite models and ontologically neutral logic. *J. Philos. Logic.*, T. 30, pp. 79–96.

Helms, L. L (1969). *Introduction to potential theory.* Wiley, N.Y.

Hervé, R. E. (1962). Recherches axiomatique sur la théorie des fonctions surharmoniques et du potentiel. *Ann. Inst. Fourier*, T. 12, pp. 415–571.

Kellog, O. (1929, 1953). *Foundation of potential theory.* Dover Publications (1953), Springer (1929).

Lagrange, J. (1773). *Oeuvres.* T. VI, p. 348, T. VII.

Landkof, N. S. (1966). *Osnovhy sovremennoy teoryi potenzyala.* Nauka Moskov.

Laplace, P. (1782). *Oeuvres de Laplace.* T. 10, p. 362, T. 11, p. 278, Paris.

Lapunov, (1898). Sur certaines questions qui se rattachent au probléme de Dirichlet. *J. Math Pures et Appl*, T. 4, Ser. 6, pp. 241–311.

Lejeune-Dirichlet, P. G. (1887). *Vorlesungen über die im umgekehrten Verhältnis des Quadrats der Entfernung wirkenden Kräfte.* Leipzig.

Lebesgue, H. (1912). Sur le probléme de Dirichlet. *C. R. Ac Sc.*, T. 154, févirier, p. 335.

Lebesgue, H. (1912). Sur les cas d'impossibilite' de probléme de Dirichlet. *C. R. Sem. Soc. Math. France*, T. 41, Nov., p. 17.

Lebesgue, H. (1914). Conditions de régularité, d'irrégularité, conditions d'impossibilité dans le problémé de Dirichlet. *C. R. Ac. ScT.*, 178, Janvier, p. 349.

Le Roux, J. (1914). Sur le probléme de Dirichlet. *J. Math. Pres et Appl.*, T. 10, pp. 189–230.

Marcinkowska, H. (1972). *Wstęp do teorii równań różniczkowych cząstkowych.* PWN, Warszawa.

Mokobodzki, G., Sibony, D. (1968). Sur une propriété characteristique des cones de potentiels. *C. R. Acad. Sci*, T. 226, pp. 215–218.

Neumann, C. (1877). *Untersuchungen über das logaritmische und newtonische Potential.* Leipzig.

Neumann, F. R. (1920). Zur Integration der Potentialgleichung vermittelst C. Neumann's Methode des arithmetischen Mittels. *Math. Ann.*, T. 55, 56.

Picard, E. (1922–28). *Traité d'analyse*, Paris.

Picard, E. (1914). Sur la distribution de l'electricite avec la loi de Neumann. *Rendic Cire. Mat.*, Palermo, T. 37, pp. 249–261.

Poisson, (1813). Remarques sur une equation qui se presente dans la théorie de l'attraction des spheroids. *Nouveau Bulletin de la Société philomatique de Paris*, T. III, pp. 388–392.

Perron, O. (1924). Eine neue Behandlung der ersten Randwertaufgabe für $\triangle u = 0$. *Math. Zeitschrift*, T. 18, H. 1, p. 2.

Prilenko, A., Sołomentzew, J. (1984). Potenzyala theorya. *Matem. Encyclop.*, T. 4, pp. 523–531 Izdat, Sov, Encycl.

Robin, (1886). Sur la distribution de l'electricité. *Ann. Ec. Norm Sup.*, T. 3, Ser. 3, suppl.

Riesz, F. (1926). Sur les fonctions subharmoniques et leur rapport ŕ la théorie du potentiel I. *Acta Math.*, T. 48 (1926), Part II Act Math. T. 54 (1930).

Steklov, (1899). Sur l'existence des fonctions fondamentales. *C. R. Ac. Sc.*, T. 128, p. 808.

Stożek, W. (1925). Sur l'allure des fonctions harmoniques au voisinage d'un point exceptionelle. *C. R. Ac. Sc.*, T. 180, Mars.

Schwartz, H. A. (1872). Zur integration des partiellen differentgleichung. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. *J. Reine Angew. Math.*, T. 74, pp. 218–253.

Tsuji, M. (1975). *Potential theory in modern function theory.* Chelsea Publ. Co. N.Y.

Wermer, J. (1974). *Potential theory.* Springer Berlin.

Wiener, N. (1924). The Dirichlet problem. *J. Math. Phys.*, T. 3, pp. 127–146.

Wiener, N. (1924). Certain notions in potential theory. *J. Math. Phys.*, T. 3, pp. 24–51.

Wilkosz, W. (1922). Sur un point fondamental de la theorie du potentiel. *C. R. Ac. Sc.*, T. 174, Fevier, p. 435.

Zaremba, S. (1897). Sur le probléme de Dirichlet. *Ann Éc. Norm. Sup.*, T. 24, pp. 251–258.

Zaremba, S. (1901). Sur les fonctions fondementales. *Bull. Ac. Sc.*, Cracovie.

Zaremba, S. (1909). Sur le principe du minimum. *Bull. Ac. Sc.*, Cracovie (Juillet, pp. 197–264).

Chapter 10

On theories and theorizing

Summary

The concept of a "theory" is one of the most fundamental concepts in science. Its interpretation and connotations vary according to the domain of science. The most formal meanings of the concept of theory are used in logic and mathematics, and the least formal ones are accepted in e.g. the arts, history or politics. The main aim of this chapter is both to present and discuss formal and non – formal definitions of the concept of theory, as well as briefly comment on the pros and cons of theorizing. In order to illustrate the usefulness of chosen formal definitions, propositions of formal definitions will be presented that may be useful in presenting theories in the fields of econometrics, economics, finance and sociology.

Keywords: theory, formal theory, non – formal theory, domain of a theory, co – domain of a theory, range of a theory, model, theorizing, modeling.

10.1. Introduction

According to the Longman Dictionary of Contemporary English (3rd ed. 1995), the word "theory" means an idea or set of ideas describing real life phenomena or the principles or rules of their evolution. Theories are assumed to be true explanations of particular phenomena. For many scientists, a theory is a system of scientifically justified laws that are logically connected and explain a large class of phenomena. There are also scientists for whom the concept of theory denotes a part of knowledge which is scientifically justified, but which is not directly practical, although may arise from practice. Originally $\theta\varepsilon\omega\rho\acute{\iota}\alpha$ signified seeing or imagining something real or abstract/virtual. It is also now used in the sense of an opinion or view about the final conclusions from an analysis of the observations made in a study. In history, politics and law, a theory means a doctrine, system, concept, or view. In turn, in the empirical sciences, a theory usually means a synthesis of particular empirical facts that take the form of refutable conclusions.

From the point of view of the practical description of a theory, any theory takes the form of a set of statements, opinions, views, ascertainments, propositions, lemmas, theorems and/or definitions expressed either verbally, symbolically or in both ways. Some theories can be very simple in their description and some very rich.

The formulation of such theories requires the use of language. Languages include natural systems of communication (e.g. English), symbolic, logical or mathematical languages and the languages of particular fields of science. It should be stressed that there are many spoken and written languages, which can be natural, scientific or constructed (e.g. Esperanto) and used separately or in combination to describe a theory by its author(s), other(s) speaker(s) or scientists.

In Section 2, we discuss the definition of the concept of a theory in the field of logic and the influence of this concept on other disciplines of science.

Section 3 presents the concepts of theories used in mathematics.

Section 4 contains a presentation and discussion of the concepts of theories in other chosen fields of science.

Section 5 gives some notes on the principles of theorizing.

10.2. Theories in logic

In logic, as well as in any other field of science, the main question is "what is a theory?", i.e. what are the elements or groups of elements which together form a theory. According to Aristotle, a theory describes our knowledge of the causes and effects of facts, events or sequences of events called phenomena. Such knowledge originates either from experience or reasoning. It takes the form of opinions, views, conclusions and descriptions of the results of experiments. The word "theory" is used in his books devoted to methods, phenomena observed in the sky, the soul, senses, motion, forms of things, matter, elements of Nature, origin of the world, features of bodies.

For our further discussion, in order to be concise, it will be useful to present the following general description of the family of theories in the field of logic:

$$\mathbb{T}^{\lambda} \equiv (\mathbb{L}^{\lambda}_{\nu},\ \mathbb{L}^{\lambda}, \mathbb{L}^{\lambda}_{o},\ \mathbb{A}^{\lambda},\ \mathbb{C}\mathrm{n}\left(\mathbb{A}^{\lambda}\right),\ \mathbb{IR}^{\lambda},\ \mathbb{DEF}^{\lambda},\ \Theta^{\lambda}\hat{\alpha}\mathbb{P}^{\lambda},\ \mathbb{EC}^{\lambda}), \tag{10.1}$$

λ denotes the field of logic and:
$\mathbb{T}^{\lambda}$ – a set or family of theories,
$\mathbb{L}^{\lambda}_{\nu}$ – the natural (ethnic) language used to describe a theory,
$\mathbb{L}^{\lambda}$ – the language of formalized logic,
$\mathbb{L}^{\lambda}_{o}$ – a chosen set of primary original terms or sentences, $\mathbb{L}^{\lambda}_{o} \subset \mathbb{L}^{\lambda}$,
$\mathbb{A}^{\lambda}$ – a set of logical axioms,
$\mathbb{C}\mathrm{n}\left(\mathbb{A}^{\lambda}\right)$ – a set of established, according to a given theory, consequences of the axioms from $\mathbb{A}^{\lambda}$,
$\mathbb{IR}^{\lambda}$ – a set of logical inference rules,
$\mathbb{DEF}^{\lambda}$ – a set of definitions,
$\Theta^{\lambda}\hat{\alpha}\mathbb{P}^{\lambda}$ – a set of basic theorems of logic and their proofs,
$\mathbb{EC}^{\lambda}$ – a set of criteria for evaluating a theory.

There are many interpretations of $\mathbb{L}^{\lambda}$. From the point of view of grammar, $\mathbb{L}^{\lambda}$ means the following (see, e.g. T. Czeżowski (1949), T. Kwiatkowski (2002), K. Ajdukiewicz (1965)):
$\mathbb{L}^{\lambda}_{1} \equiv$ (Nouns, verbs, adjectives, adverbs, pronouns, subjects, predicates, grammatical patterns, prepositions, conjunctions, sentences, style, aims of $\mathbb{L}^{\lambda}$).
Other meanings are:
$\mathbb{L}^{\lambda}_{2} \equiv$ (communication system of a given nation of people which uses spoken or written words, expressions, symbols, signals, codes or other signs or semantic instructions),
$\mathbb{L}^{\lambda}_{3} \equiv$ ($\mathbb{L}$ – vocabulary, $\mathbb{L}$ – syntactic instructions, $\mathbb{L}$ – semantic instructions),
$\mathbb{L}^{\lambda}_{4} \equiv$ ($\mathbb{L}$ – rules governing signs, $\mathbb{L}$ – use of signs, $\mathbb{L}$ – syntactic rules, $\mathbb{L}$ – semantic rules, expressions, sentences, style of speaking or writing),
$\mathbb{L}^{\lambda}_{5} \equiv$ (system of signs, system of syntactic rules, system of semantic rules, system of $\mathbb{L}$ – relations between $\mathbb{L}$ – users).

Note 10.1. In describing their theories, logicians are careful to use only a very small range of the tools available in the given languages $\mathbb{L}^{\lambda}_{i}$, $i = 1, 2, \ldots, 5$. Very often, they restrict the meaning of a commonly used word with several meanings to their intended meaning specific to the field of logic. For these reasons, objects from $\mathbb{L}^{\lambda}_{i}$, $i = 3,\ 4,\ 5$ are preferred to objects from $\mathbb{L}^{\lambda}_{1}$ and $\mathbb{L}^{\lambda}_{2}$.

For the purposes of logical analysis, logicians are forced to simplify the meanings of words by using interpretations specific to the field of logic. This leads to suitable definitions of the language of logic, $\mathbb{L}^{\lambda}$ (see, e.g. A. Tarski (1995)). According to Tarski, the language $\mathbb{L}^{\lambda}$ may be represented as either

$$\mathbb{L}_1^{\lambda} = \left(\mathbb{A},\ \mathbb{F}_{\mathbb{A}},\ \mathbb{C}\mathrm{n}\left(\mathbb{F}_{\mathbb{A}}\right)\right),$$

where:
$\mathbb{A}$ denotes a finite or countable set of uniquely defined words (alphabet) and elements of words,
$\mathbb{F}_{\mathbb{A}}$ proper infinite set of all subsets (finite sequences of words) from $\mathbb{A}$,
$\mathbb{C}\mathrm{n}\left(\mathbb{F}_{\mathbb{A}}\right)$ is the set of monotonic, reflexive, transitive, compact consequences of sentences belonging to $\mathbb{F}_{\mathbb{A}}$,
or as

$$\mathbb{L}_2^{\lambda} = \left(\mathbb{A},\ \mathbb{F}_{\mathbb{A}},\ \mathbb{C}\mathrm{n}\left(\mathbb{F}_{\mathbb{A}}\right),\ \neg,\ \Rightarrow\right),$$

where $\neg$, $\Rightarrow$ denote the logical negation and implication connectives.

It is implicitly assumed that $\mathbb{C}\mathrm{n}\left(\mathbb{F}_{\mathbb{A}}\right)$ is obtained by the use of appropriate syntactic and semantic rules which are respected and used in formal or mathematical logic, as well as in classical or non – classical logic of sentence calculus.

Note 10.2. Advanced modern languages $\mathbb{L}^{\lambda}$ cover such new fields of logic as: paraconsistent logic, modal logic, deontic logic, temporal logic, intuitionistic logic and multivalued logic. This means that precise definitions of these languages need to specify suitable symbols of connectives, specialized terms, expressions and concepts that satisfy the assumed postulates of a description of the language of logic.

Many axioms of logic $\mathbb{A}^{\lambda}$ have been applied. Among these, we highlight the axioms of predicate calculus, i.e. the sentence formulas which originate from tautologies of sentence calculus. The following are particularly relevant tautologies (called sometimes laws): identity, separation, double negation, non-contradiction, contraposition, syllogism, de Morgan's laws, overfillment, distributiveness with respect to disjunction (alternatives) or conjunction.

These axioms, names and rules of sentence calculus applied to objects from $\mathbb{L}^{\lambda}$ corresponding to a particular field of logic enable us to fix the set of $\mathbb{C}\mathrm{n}\left(\mathbb{A}^{\lambda}\right)$, i.e. the logical consequences of $\mathbb{A}^{\lambda}$, or if $\mathbb{A}^{\lambda}$ also contains particular further assumptions, the consequences of these assumptions. In order to establish such consequences, particular forms of the set $\mathbb{IR}^{\lambda}$ of rules for deduction, induction, reduction and abduction by logical inference are supported by suitable definitions, the fundamental theories of logic and methods of proof, i.e. elements of the sets $\mathbb{DEF}^{\lambda}$ and $\Theta^{\lambda}\hat{\alpha}\mathbb{P}^{\lambda}$. Any given theory in logic, $\mathbb{T}^{\lambda}$, should be evaluated. Possible criteria, $\mathbb{EC}^{\lambda}$, for evaluating a theory are: non-contradictoriness, connectivity, refutability, conclusiveness, simplicity, range of applicability (either theoretical or practical), degree of originality, degree of potential influence on other theories, completeness with respect to axioms and assumptions, completeness with respect to known rules of inference, completeness with respect to existing logical definitions, compactness of $\Theta^{\lambda}\hat{\alpha}\mathbb{P}^{\lambda}$, degree of asymmetry between the number of axioms and assumptions and the number of theorems and consequences.

Note 10.3. This proposed abstract scheme for $\mathbb{T}^{\lambda}$ concerns the formal representation of any theory in logic. The presentation of a particular theory or theories of logic takes the form of articles, monographs, textbooks, conference papers and/or lectures. They generally do not adopt the above schematic form of describing $\mathbb{T}^{\lambda}$. However, it is possible to reconstruct, from the written forms of particular theories of logic mentioned above, the elements of $\mathbb{L}_{\nu}^{\lambda}$, $\mathbb{L}^{\lambda}$, $\mathbb{L}_{o}^{\lambda}$, $\mathbb{A}^{\lambda}$, $\mathbb{C}\mathrm{n}\left(\mathbb{A}^{\lambda}\right)$, $\mathbb{IR}^{\lambda}$, $\mathbb{DEF}^{\lambda}$, $\Theta^{\lambda}\hat{\alpha}\mathbb{P}^{\lambda}$ and $\mathbb{EC}^{\lambda}$ and write them (or list them) in succession under the nine constructs given immediately above. Such a means of reconstructing a theory seems to be boring for some readers or researchers and too oriented towards computer search algorithms, but may have some advantages when one wants to use elements of $\mathbb{EC}^{\lambda}$ to evaluate a specific theory in comparison to other theories.

Note 10.4. A logical theory $\mathbb{T}_{i}^{\lambda}$ formulated in the languages $(\mathbb{L}_{\mathrm{j}}^{\nu}, \mathbb{L}_{j}^{\lambda}, \ \mathbb{L}_{o,j}^{\lambda})$ should be consistent, complete or non-complete, and contain a set of sentences which enable to formulate a model of the theory. This theory may be aimed at formulating new axioms and assumptions or consequences, connections and relations, as well as at discussing their relations to new theorems and methods of proof. In this respect, special attention is devoted to the theories of parainconsistencies and their role in metalogical and logical studies of scientific opinions, intuitions and beliefs, both in formal and informal domains of investigation (see, e.g J. Perzanowski (2001)) .

10.3. Theories in mathematics

Accepting the view that mathematical logic belongs to the fields of both formal logic and mathematics, the scheme 10.1 from §10.2 is also suitable for defining and analyzing a concrete theory in mathematical logic. Thus the scheme 10.1 takes the form:

$$\mathbb{T}^{m.\lambda} \equiv (\mathbb{L}_{\nu}^{m},\ \mathbb{L}^{m.\lambda}, \mathbb{L}_{o}^{m.\lambda},\ \mathbb{A}^{m.\lambda},\ \mathbb{C}\mathrm{n}\left(\mathbb{A}^{m.\lambda}\right),\ \mathbb{IR}^{m.\lambda},\\ \mathbb{DEF}^{m.\lambda},\ \Theta^{m.\lambda}\hat{\alpha}\mathbb{P}^{m.\lambda},\ \mathbb{EC}^{m.\lambda}), \quad (10.2)$$

where $\mathbb{L}_{\lambda}^{m}$ is a substructure of a natural language used in mathematical theories, $m.\lambda$ denotes mathematical logic and the other symbols have the same meaning as above, except for the restriction that all designates and connotations correspond to objects of mathematical logic.

For theories from other fields of mathematics (e.g. set theory, topology, functional analysis, mathematical analysis, algebra, group theory, geometry, probability theory, mathematical statistics, numerical analysis, theory of algorithms), we need to respecify the scheme 10.2 to the following form:

$$\mathbb{T}^{m} \equiv (\mathbb{L}_{\nu}^{m},\ \mathbb{L}^{m}, \mathbb{L}_{o}^{m},\ \mathbb{A}^{m},\ \mathbb{C}\mathrm{n}\left(\mathbb{A}^{m}\right),\ \mathbb{IR}^{m},\ \mathbb{DEF}^{m},\ \Theta^{m}\hat{\alpha}\mathbb{P}^{m},\ \mathbb{EC}^{m}), \quad (10.3)$$

where the superscript m here denotes any given field of mathematics or subsection of such a field.

The list of concrete objects considered within the nine constituent constructs $\mathbb{L}_{\nu}^{m}, \ldots, \mathbb{EC}^{m}$ in 10.3 would be m – specific, so this list would not only change according to m, but also according to the mathematical problem studied. For instance, when m denotes mathematical analysis, the problems studied may be: the existence of solutions to a particular ODE (ordinary differential equation) or PDE (partial differential equation), or the particular type of stability of such a solution. In order to denote this change of description according to the group of problems to be solved, we may adapt the notation adopted in 10.3 to:

$$\mathbb{T}_{i}^{m} \equiv (\mathbb{L}_{\nu}^{m},\ \mathbb{L}_{i}^{m}, \mathbb{L}_{o,i}^{m},\ \mathbb{A}_{i}^{m},\ \mathbb{C}\mathrm{n}_{i}\left(\mathbb{A}_{i}^{m}\right),\ \mathbb{IR}_{i}^{m},\ \mathbb{DEF}_{i}^{m},\ \Theta_{i}^{m}\hat{\alpha}\mathbb{P}_{i}^{m}\ ,\ \mathbb{EC}_{i}^{m}), \quad (10.4)$$

where "i" denotes a given problem or group of problems to be solved by theory (or theories) $\mathbb{T}_{i}^{m}$.

It should be mentioned that within a given field of mathematics (fixed m), the changes in $\mathbb{L}_{i}^{m}, \ldots, \mathbb{EC}_{i}^{m}$ are small when we change $i \equiv i(m)$ compared to the change in the language used in different fields.

Note 10.5. When evaluating a given $\mathbb{T}_{i}^{m}$ for some $(m,\ i)$ and comparing it to theory $\mathbb{T}_{i'}^{m}$, $i \neq i'$, where i' corresponds to an extension of the theory $\mathbb{T}_{i}^{m}$, the question of measuring or evaluating the difference between these theories, $diff(\mathbb{T}_{i}^{m},\ \mathbb{T}_{i'}^{m})$, arises. Formally, such a difference would exist if and only if at least one of the following inequalities hold: $\mathbb{L}_{i}^{m} \neq \mathbb{L}_{i'}^{m}, \ldots,$

$\mathbb{A}_i^m \neq \mathbb{A}_{i'}^m$, ..., $\mathbb{EC}_i^m \neq \mathbb{EC}_{i'}^m$, where inequality occurs when at least one element (object) differs in corresponding groups of objects, e.g. in the list of objects of $\mathbb{A}^m$ in theories i and i'. We assume that important differences, in terms of the development of the field m of mathematics, occur if we observe differences within $\mathbb{A}^m$, $\mathbb{C}\mathrm{n}\,(\mathbb{A}^m)$, $\Theta^m \hat{\alpha} \mathbb{P}^m$ or $\mathbb{L}_o^m$.

Note 10.6. Due to the commercial and industrial applications of applied mathematics, probability, statistics, econometrics, biometrics, technometrics, algorithmics and informatics, there are natural incentives to make advances in purely mathematical theories, such as topology, analysis, algebra, group theory and differential geometry. The diffusion of ideas between disciplines has brought modern mathematical theories of entropy (which originated in physics) into the fields of topology, statistics, operator theory, and chaos theory to be applied in physics, and next into the field of ODEs, PDEs, nonlinear algebra, numerical analysis, statistics and informatics. In turn, theories of dynamic systems are now being used in physics, biology, the engineering sciences, astrophysics and economics.

It should be stressed that we may attach numerical characteristics to each of the nine groups of mathematical objects considered, which is useful in describing theories. However, the structural positions within and between groups are much more important, especially in the context of the properties of their links, types of dependencies, degrees of interdependence or independence. In these respects, the most influential aspects are the properties of the structural connections between the elements of $\mathbb{A}^m$, $(\mathbb{A}_i^m)_1^n$, $\mathbb{C}\mathrm{n}\,(\mathbb{A}^m)$, $\mathbb{C}\mathrm{n}\,(\mathbb{A}_i^m)_1^n$, $\mathbb{DEF}^m$, $(\mathbb{DEF}_i^m)_1^d$, $\Theta^m \hat{\alpha} \mathbb{P}^m$ and $(\Theta_i^m \hat{\alpha} \mathbb{P}_i^m)_1^p$.

Moreover, as M. Lange (2013) states, the use of mathematical theory in practice to explain physical facts enables the explanandum to be more precisely described than by ordinary causal laws. This means that there exist scientific explanations based on such facts which are modally stronger than any known ordinary causal statements. Such explanations are applied in theoretical physics until they are refuted by theoretical or empirical results from an experimentum crucis.

10.4. Theories in some other fields of science

Historically, physics has been the discipline which has most often used mathematical tools in formulating its theories and applications. New discoveries in theoretical physics have often been strictly connected with the discovery of suitable new theories in pure or applied mathematics and these discoveries were often made by the same person (e.g. I. Newton, J. d'Alembert,

J. Lagrange, W. Hamilton, P. Laplace, J. Fourier, P. Dirac, S. Poisson). The saturation of, not only, theoretical physics with mathematical and logical concepts, as well as the tools of inference is continuously growing, so neither professional mathematicians nor physicists can see the demarcation lines between these two disciplines of science. The following are important problems in discussing the essence and role of theories in the field of physics:

p1) can mathematical models describe, explain and help to predict physical phenomena well?

p2) what is the physical essence of observed objects and their interrelationships?

p3) what makes theories from the field of physics so popular in other fields of science?

There are many articles and books considering these problems. Some remarks will be given after presenting and discussing a formal structure for defining theories in the field of physics. It is similar in structure to the description of theories in the fields of mathematics and logic, although it differs according to the content of the elements included.

The form of this structure is as follows:

$$\mathbb{T}^{ph} \equiv (\mathbb{L}_{\nu}^{ph},\ \mathbb{L}^{ph}, \mathbb{L}_{o}^{ph},\ \mathbb{A}^{ph},\ \mathbb{C}\mathrm{n}\left(\mathbb{A}^{ph}\right),\ \mathbb{IR}^{ph},\ \mathbb{DEF}^{ph},\ \Theta^{ph}\hat{\alpha}\mathbb{P}^{ph},\ \mathbb{EC}^{ph}), \tag{10.5}$$

where:
$\mathbb{L}_{\nu}^{ph}$ is a substructure of a natural language used in physics, $ph \equiv$physics, $\mathbb{L}^{ph}$, $\mathbb{L}_{o}^{ph}$, $\mathbb{A}^{ph}$, $\mathbb{C}\mathrm{n}\left(\mathbb{A}^{ph}\right)$, $\mathbb{IR}^{ph}$, $\mathbb{DEF}^{ph}$ are groups of specified objects belonging to the language of physics, the set of axioms and assumptions used in physics, a list of the direct consequences of these axioms and assumptions, a list of the rules of inference and definitions that are used in given theories from the field of physics, a list of theorems and their proofs if they are not conclusions from the axioms of physics, but inspired by empirical facts or facts implied by theories not from the field of physics. The lists of elements of $\mathbb{EC}^{ph}$ may be different or algorithmically similar to the list of elements of $\mathbb{EC}^{\lambda}$ or $\mathbb{EC}^{m}$.

The formal structure 10.5 embraces any particular theory from the field of physics, $\mathbb{T}_{i}^{ph}$, where "i" is a subscript corresponding to the chosen theory. Thus the natural number $i \in \mathbb{N}$ may correspond to the theory of e.g. motion, light, gravity, inertia, entropy, temperature, equilibrium, fluids, solids, matter, relativity or elementary particles. Redefining 10.5 with respect to a fixed $i \in \mathbb{N}$ gives the following definition of the theory $i \in \mathbb{N}$:

$$\mathbb{T}_i^{ph} \equiv (\mathbb{L}_{i,\nu}^{ph},\ \mathbb{L}_i^{ph}, \mathbb{L}_{o,i}^{ph},\ \mathbb{A}_i^{ph},\ \mathbb{C}\mathrm{n}_i\left(\mathbb{A}_i^{ph}\right),\ \mathbb{IR}_i^{ph},\ \mathbb{DEF}_i^{ph},\ \Theta_i^{ph}\hat{\alpha}\mathbb{P}_i^{ph},\ \mathbb{EC}_i^{ph}), \tag{10.6}$$

where the subscript $i \in \mathbb{N}$ implies that the included lists of objects usually contain less elements than their counterparts in 10.5, i. e. the lists $\mathbb{L}_{i,\nu}^{ph}$, $\mathbb{L}_i^{ph}$, $\ldots$, $\mathbb{EC}_i^{ph}$ are shorter than the lists $\mathbb{L}_\nu^{ph}$, $\mathbb{L}^{ph}$, $\ldots$, $\mathbb{EC}^{ph}$.

Example. The author of this text does not have access to the original version of Newton's theory of gravitation and therefore only sketches how to formulate some of the elements belonging to chosen lists of objects of $\mathbb{L}_{0,1}^{ph}$, $\mathbb{A}_1^{ph}$ describing this theory.

Thus we may describe the following lists with particular groups of objects:

$\mathbb{L}_{0,1}^{ph} \equiv$ (time, space, mass, force, acceleration, distance between bodies, gravity, impetus, external force, system of bodies, centre of mass, equation);

$\mathbb{A}_1^{ph} \equiv$ (time is continuously and uniformly changing; space is 3-dimensional, isotropic, homogenous and continuous; each planet moves on an almost eliptic orbit around the sun, which is motionless; any body is either at rest or moves uniformly and rectilinearly till the actions of another body changes this state of motion; the rate of change of a body's impetus is equal to the force acting on this body; forces interacting on themselves are equal as to their absolute values, but act in opposite directions);

$\mathbb{C}\mathrm{n}_1\left(\mathbb{A}_1^{ph}\right) \equiv$ (the gravitational force between two objects is proportional to the product of the masses of the two bodies divided by the square of the distance between them, such that the constant of proportionality is the gravitational constant, the force of inertia of a body is equal to the negative of the product of the mass of that body and the vector of motion or convection);

$\mathbb{IR}_1^{ph} \equiv$ ($\mathbb{IR}^\lambda$, $\mathbb{IR}^m$, the rules of carrying out and analyzing physical nature experiments);

$\mathbb{DEF}_1^{ph} \equiv$ (definitions of impetus, force, acceleration, gravitational force, force of inertia).

Note 10.7. As the above example shows, a full reconstruction of all the lists of the objects of theory $\mathbb{T}_1^{ph}$ by "manual" search would be very time consuming. It would be preferable to use specialist computer programs for sorting and searching, possibly enriched with an additional procedure (written by a professional programer) that would carry out automatic searching and sorting, as well as printing the results of a synthetic description and evaluation of $\mathbb{T}_1^{ph}$. Similar comments are valid for any $\mathbb{T}_i^{ph}$, $\mathbb{T}_i^m$, $\mathbb{T}_i^\lambda$.

In order to broaden our view of the content of such concepts of scientific theories, we now consider a structure for describing econometric theories. Econometrics is a discipline which plays a similar role in economics as theoretical physics plays within physics.

The mail subfields of econometrics, as are easily seen from the volumes of the journal "Econometrica", founded in the year 1932, are: mathematical economics, economic theory, economic statistics, theory of organization, operations research, economic game theory, statistical econometrics, econometric models and methods, the theory of forecasting, the application of econometrics in describing, explaining and forecasting economic phenomena, as well as in economic decision making. Due to the successful PR-activities of statisticians in developed countries, which were reforged into vast educational programs and skillfully created needs for the application of statistical methods in almost all human activities, econometrics has also been pushed into the direction of statistical econometrics. This, in turn, has yielded a crop in the form of the almost entirely statistically oriented "Journal of Econometrics" which has two series: theoretical and applied. It is hoped that the above comments may help the readers to imagine the kinds of interrelationships existing between possible objects of the set of econometric theories $\mathbb{T}^e$, their ontological status, the types of languages used, types of axioms and their consequences, types of rules for reasoning or definitions, kinds of theorems which do not result from the axioms and their proofs, and finally, kinds of criteria for evaluating theories. Thus, in the case of econometric theories, the structure used to define such a theory is as follows:

$$\mathbb{T}^e \equiv (\mathbb{L}^e_\nu,\ \mathbb{L}^e, \mathbb{L}^e_o,\ \mathbb{A}^e,\ \mathrm{Cn}\left(\mathbb{A}^e\right),\ \mathbb{IR}^e,\ \mathbb{DEF}^e,\ \Theta^e\hat{\alpha}\mathbb{P}^e,\ \mathbb{EC}^e), \qquad (10.7)$$

where the superscript "e" stands for "econometric", and the sets of elements (objects) belonging to the groups of elements denoted by, appropriately, $\mathbb{L}^e_\nu$, $\mathbb{L}^e$, ..., $\mathbb{EC}^e$ are specific to econometric theories. When we take only one specific theory number "i ≡e.g. the theory of linear single-equation standard econometric models", the structure 10.7 reduces to the following descriptive structure:

$$\mathbb{T}^e_i \equiv (\mathbb{L}^e_{i,\nu},\ \mathbb{L}^e_i, \mathbb{L}^e_{o,i},\ \mathbb{A}^e_i,\ \mathrm{Cn}_i\left(\mathbb{A}^e_i\right),\ \mathbb{IR}^e_i,\ \mathbb{DEF}^e_i,\ \Theta^e_i\hat{\alpha}\mathbb{P}^e_i,\ \mathbb{EC}^e_i), \qquad (10.8)$$

where the collections (sets) $\mathbb{L}^e_{i,\nu}$, ..., , $\mathbb{EC}^e_i$ are specific to theory "i". The following are some basic elements of the set $\mathbb{L}^e_{o,i}$:
linear, single equation, statistical, econometric, model; vector, vector of parameters, structural parameters, stochastic structural parameters, matrix, data, numerical data matrix, random variable, random vector, probability

distribution, normal distribution, rank of a matrix, probability space, set of elementary events, Borel σ – field of subsets of the set of elementary events, probability measure, set of real numbers, Euclidean space, expected value, dispersion matrix (variance – covariance matrix), nonsingular matrix, full column rank matrix, non – collinearity, correlation matrix, estimator, unbiased estimator, admissible estimator, efficient estimator, consistent estimator, asymptotically unbiased estimator, predictor, unbiased predictor, efficient predictor, robust estimator, robust predictor.

The following is one way of describing a model of $\mathbb{T}_i^e$ from 10.8 (see Milo (1985)):

$$MO(\mathbb{T}_i^e) \equiv (\mathbb{R}^{n\times k},\ S_u \equiv (\Omega,\ \mathcal{F}_\Omega, \mu)\,,\ \mathbb{Y} = \mathbb{x}\boldsymbol{\beta} + \boldsymbol{\Xi}, \boldsymbol{\beta} \in \mathbb{R}^k, \mathbb{x} \in \mathbb{R}^{n\times k},$$
$$\rho\,(\mathbb{x}) = \mathrm{k},\ \mathrm{D}\,(\boldsymbol{\Xi}) = \mathrm{D}\,(\mathbb{Y}) = \sigma^2\mathbb{I}_n,\ P_{\boldsymbol{\Xi}} \sim N(\overrightarrow{0}, \sigma^2\mathbb{I}_n)),$$

where:
$\mathbb{R}^{n\times k}$ is the real Euclidean $n \times k$ dimensional space,
S_u is a probability space where Ω is the set of elementary events, $\mathcal{F}_\Omega$ is the Borel σ – field of subsets of Ω, μ is a probability measure defined on $\mathcal{F}_\Omega$,
$\mathbb{Y}$ is a random $n \times 1$ vector which is equal to the sum of the product of the data matrix $\mathbb{x}$ and the $\boldsymbol{\beta}$ vector of real unobserved parameters, and a random vector $\boldsymbol{\Xi}$ of errors specified according to $MO(\mathbb{T}_i^e)$, ρ denotes the column rank of matrix $\mathbb{x}$, $D(\mathbb{Z})$ denotes the dispersion matrix of random vector $\mathbb{Z}$, σ^2 is the variance of each coordinate of vector $\boldsymbol{\Xi}$ or $\mathbb{Y}$, $\mathbb{I}_n$ is the $n \times n$ identity matrix, $P_{\boldsymbol{\Xi}}$ denotes the probability distribution measure for vector $\boldsymbol{\Xi}$, which is assumed here to be the probability measure for a Gaussian normal distribution with mean equal to an $n \times 1$ vector of zeros, and the variance-covariance matrix equal to the matrix $\sigma^2\mathbb{I}_n$.

The set $\mathbb{A}_i^e$ of axioms and assumptions of $\mathbb{T}_i^e$ contains, among other things, the following elements:

- a probability measure μ fulfilling the axioms:

  - ⧫ for each random event $E \in \mathcal{F}_\Omega :\ \mu\,(E) \geq 0$,
  - ⧫ if $\{E_l\}$ is a countable subset of pairwise disjoint sets belonging to the Borel σ – field $\mathcal{F}_\Omega$, then $\mu\,(\bigcup_l E_l) = \sum_l P(E_l)$,
  - ⧫ $\mu\,(\Omega) = 1$.

It is assumed that:

- μ is a real valued set function with domain $\mathcal{F}_\Omega$ and codomain $[0,\ 1]$,
- the dimension of the results of the observation space (i.e. "n") is greater than the dimension of the column space of the k explanatory factors observed, i. e. $n > k$,
- column $rank(\mathbb{x}) \ = \ rank(\mathbb{x}^T\mathbb{x}) = k < n$,
- the fixed real matrix $\mathbb{x}$ is well conditioned numerically, and is not random,
- the results of the observations $(\mathbb{Y},\ \mathbb{x})$ are not atypical, and $\mathbb{y}$ is generated according to the model $MO(\mathbb{T}_i^e)$, which is treated as a Data Generating Process (DGP),
- for fixed n, k and $\mathbb{x}$, the quadratic form $\varphi(\boldsymbol{\beta}) \equiv (\mathbb{Y} - \mathbb{x}\boldsymbol{\beta}, \mathbb{Y} - \mathbb{x}\boldsymbol{\beta})$ is continuous and differentiable with respect to $\boldsymbol{\beta}$, i. e. $\frac{\partial\varphi(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}}$ exists,
- $est\,(\boldsymbol{\beta}) \equiv \mathbb{B} = \left(\mathbb{x}^T\mathbb{x}\right)^{-1}\mathbb{x}^T\mathbb{Y} \neq \overrightarrow{0}$, $\boldsymbol{\beta} \neq \overrightarrow{0}$ and $\left(\mathbb{x}^T\mathbb{x}\right)^{-1}$ exists since $\rho = k$,
- the predictor $\widehat{\mathbb{Y}}$ of $\mathbb{Y}$ is non – zero in value, and $\widehat{\mathbb{Y}} = \mathbb{x}\mathbb{B} = \left(\mathbb{x}^T\mathbb{x}\right)^{-1}\mathbb{x}^T\mathbb{Y}$,
- the estimator $\mathbb{E} = est\,(\boldsymbol{\Xi}) = \left(\mathbb{I} - \mathbb{x}\left(\mathbb{x}^T\mathbb{x}\right)^{-1}\mathbb{x}^T\right)\mathbb{Y} \neq \overrightarrow{0}$,
- the estimator $\hat{\sigma}^2 = est\left(\sigma^2\right) = \frac{\mathbb{E}^T\mathbb{E}}{n-k}$, $0 < \hat{\sigma}^2 < \infty$,
- $\mathbb{x}$ is fixed and known from the observations, although its DGP is unknown,
- $\mathbb{B}$, $\widehat{\mathbb{Y}}$, $\mathbb{E}$ have non-singular multivariate Gaussian distributions,
- the axioms and assumptions of arithmetic and linear algebra for the set of real numbers are satisfied.

Some of the consequences of the above axioms and assumptions, which belong to the set $\mathbb{C}\mathrm{n}\,(\mathbb{A}_i^e)$, are as follows:

- the estimator $\mathbb{B}$ is, under $MO(\mathbb{T}_i^e)$, efficient in the sense of being BLUE (best linear unbiased estimator), consistent, sufficient and Gaussian,
- the predictor $\widehat{\mathbb{Y}}$ is unbiased, efficient, consistent and Gaussian such that
  $MSE(\widehat{\mathbb{Y}}) < MSE(\mathbb{Y})$ and $var\left(\widehat{\mathbb{Y}}^T\widehat{\mathbb{Y}}\right) < var(\mathbb{Y}^T\mathbb{Y})$, where MSE denotes mean square error.
- the estimator $\mathbb{E}$ is BLUE and the values of the MSE for $\mathbb{Y}$ and $\mathbb{B}$, as well as $var(\mathbb{Y}^T\mathbb{Y})$ and $var(\mathbb{B}^T\mathbb{B})$ do not depend on k,
- if $n = k$, then $MSE\,(\mathbb{E}) = expectation\left(\mathbb{E}^T\mathbb{E}\right) = var\left(\mathbb{E}^T\mathbb{E}\right) = 0$,
- the random vectors $\mathbb{Y}$, $\widehat{\mathbb{Y}}$, $\mathbb{B}$ and $\mathbb{E}$ have multivariate non – singular distributions,
- $\boldsymbol{\Xi}^T\boldsymbol{\Xi}$, $\mathbb{E}^T\mathbb{E}$, $\mathbb{Y}^T\mathbb{Y}$, and $\widehat{\mathbb{Y}}^T\widehat{\mathbb{Y}}$ when divided by σ^2 have χ^2 distributions.

The consequences presented above are important statistically and econometrically. In economic applications of $MO(\mathbb{T}_i^e)$, by $\mathbf{Y}$ it is meant, f.ex. GDP, GNP, investment, consumption, unemployment, growth of GDP quantities in time period, and by $\mathbf{X}$ it is meant a given matrix of statistical sample data for a fixed economic or other factors determining the values of $\mathbf{Y}$. Usefulness of the above listed consequences for economic or political decision making is easily foreseen.
The collection $\mathbb{IR}_i^e$ may contain, among other things:

- the rules of arithmetic and algebraic operations,
- the rules of differential calculus,
- the rules of such statistical testing procedures as the Gosset test, Fisher – Snedecor test, χ^2 – test, Durbin – Watson test, Jarque – Berry test, Dickey – Fuller test,
- the rules of data “mining”,
- the rules of interpreting data,
- the rules and theory of data modelling.

The following are particular theorems from $\Theta_i^e \hat{\alpha} \mathbb{P}_i^e$:

- the estimators $\mathbb{B}$ and $\mathbb{E}$ and the predictor $\widehat{\mathbb{Y}}$ are not robust against the translation and scaling of vectors,
- $\mathbb{B}^T\mathbb{B}$ does not have a χ^2 distribution,
- the Legendre – Gauss estimator $\mathbb{B}$ is predictively perfect when either the explanatory variables and $\boldsymbol{\Xi}$ change monotonically in time or they behave suitably symmetrically in time,
- when $\mathbb{x}$ is not random, there is no probabilistic interpretation for $corr(\mathbb{x},\ \mathbb{x})$.

The following are possible elements of the collection $\mathbb{EC}_i^e$:

- distance $dist(\mathbb{Y},\ \widehat{\mathbb{Y}})$,
- statistical significance of derived models of $\mathbb{T}_i^e$,
- suitable measures of fit given a particular collection of determinants of $\mathbb{Y}$ are inferred to be causes,
- degree of uncertainty of $\mathbb{T}_i^e$,
- rules of logical reasoning.

10.5. Final remarks

This discussion about different groups (structures) of elements, which together give a full description of theories in the fields of logic, mathematics, physics and econometrics, has focused on descriptions, explanations and examples of elements of chosen structures, which are ingredients in the definition of specific theories. It can be easily seen that these characterizations were neither exhaustive nor full. Moreover, no comparisons were made between the various theories discussed. Intuitively, it might be said, that, e.g. many elements of groups of $\mathbb{T}^\lambda$ theories from the field of logic should be used in theories developed in mathematics, physics, chemistry, biology, economics, statistics and econometrics. However, there are mathematicians and physicists, even those who work in applied fields of their disciplines, who pretend that they do not need logical calculus, intuition or assessments of their theoretical results made by researchers working in other fields of knowledge, e.g. in chemistry, biology, medicine, economics, econometrics or statistics. Such researchers verify the usefulness of models from the fields of mathematics and physics used as a means to describe objects or phenomena in their own disciplines. Hence, the following question arises: what are the principles of good theorizing?

The author shares the view that good theorizing involves formulating and verifying theory with respect to fulfillment of the following criteria:

c1) non – contradiction (within a given group of elements of a theory and between groups of elements of theories),
c2) decidability of $\mathbb{C}\mathrm{n}\,(\mathbb{A})$, $\Theta\hat{\alpha}\mathbb{P}$,
c3) completeness of a theory,
c4) range of application of $\mathbb{C}\mathrm{n}\,(\mathbb{A})$, $\Theta\hat{\alpha}\mathbb{P}$,
c5) sufficiency of a theory to describe and explain real phenomena,
c6) quality of the predictions of a theory,
c7) degree of originality of a theory,

c8) robustness of a theory to refutation,
c9) degree of abstraction in a theory,
c10) the ease with which models can be generated under a given theory.

This list of criteria is not full. However even these criteria are difficult to apply if a researcher does not possess suitably defined, preferably numerical, measures to assess a theory according to these ten criteria. According to the author, if such computable measures are included in an appropriate search and evaluation program, our, at present, comparatively scant knowledge about concrete theories found in the vast literature on any subject will take a leap forward.

Another approach to evaluating and comparing theories is connected with theorems, in particular those which are strictly formulated by logicians such that each discussed formal theory can be interpreted as an infinite set of models based on it. Thus, we may study and evaluate theories by studying and evaluating suitably chosen representative models based on them. One such model, the econometric model $MO(\mathbb{T}_i^e)$, was presented and discussed in the previous section. We may propose criteria for assessing and comparing models using a similar approach to the one based on criteria (c1) – (c10). Only (c10) requires a minor change in form by replacing "models can be generated under a theory" by "models together with submodels…". The literature cited below is recommended to readers interested in extending these ideas and views.

Bibliography

Ayer, A. J. (2000). *Language, Truth and Logic.* Penguin Books, London.

Achinstein, P. (1968). *Concepts of science.* John Hopkins Press, Baltimore.

Aristotle (1983). *Books on Metaphysics, the Heavens, Logic*, PWN, Warsaw.

Ajdukiewicz, K. (1965). *Język i poznanie.* t. II, PWN, Warsaw.

Barwise, J. (ed.) (1982). *Handbook of mathematical logic.* Amsterdam.

Bunge, M. (1973). *Method, Model and Matter.* D. Reidel Publ., Dordrecht.

Carnap, R. (1958). *Introduction to Symbolic Logic and its Applications.* Dover, N.Y.

Grzegorczyk, A. (1957). *Zagadnienie rozstrzygalności.* Warsaw.

Hempel, C. G. (1969). *On the Structure of Scientific Theories.* The Isenberg Memorial Lecture Series, 1965–66, pp. 11–38, Michigan State Univ. Press.

Lange, M. (2013). What makes a scientific explanation distinctively mathematical? *Brit. J. Phil. Sci.*, 64, pp. 485–511.

Łukasiewicz, J. (1957). *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic.* OUP Oxford.

Milo, W. (1985). *Liniowe modele statystyczne.* Wyd. UŁ, Łódź.

Popper, K. (1963). *Conjectures and Refutations.* Routledge and Kegan Paul, London.

Przełęcki, M. (1969). *The Logic of Empirical Theories.* Routledge and Kegan Paul, London.

Rao, C. R. (1965). *Linear Statistical Inference and its Applications.* Wiley, N.Y.

Ramsey, F. (1950). *The Foundations of Mathematics.* The Humanities Press, NY.

Rasiowa, H., Sikorski, R. (1968). *The Mathematics of Metamathematics.* PWN, Warsaw.

Rutkowski, A. (1978). *Elementy logiki matematycznej.* WSiP, Warsaw.

Shoenfield, J. R. (1967). *Mathematical logic.* Addison-Wesley.

Suppe, F. (1972). Theories, their Formulations and the Operational Imperative. *Synthese*, pp. 129–164.

Tarski, A. (2001). *Pisma logiczno-filozoficzne.* Tom 2, Wydawnictwo Naukowe PWN, Warsaw.

Watkins, J. (1984). *Science and Scepticism.* PUP, Princeton.

Wojciechowska, A. (1979). *Elementy logiki i teorii mnogości.* PWN, Warszawa.

Wójcicki, R. (1974). *Metodologia formalna nauk empirycznych.* Wyd. Ossolineum, PAN, Wrocław.