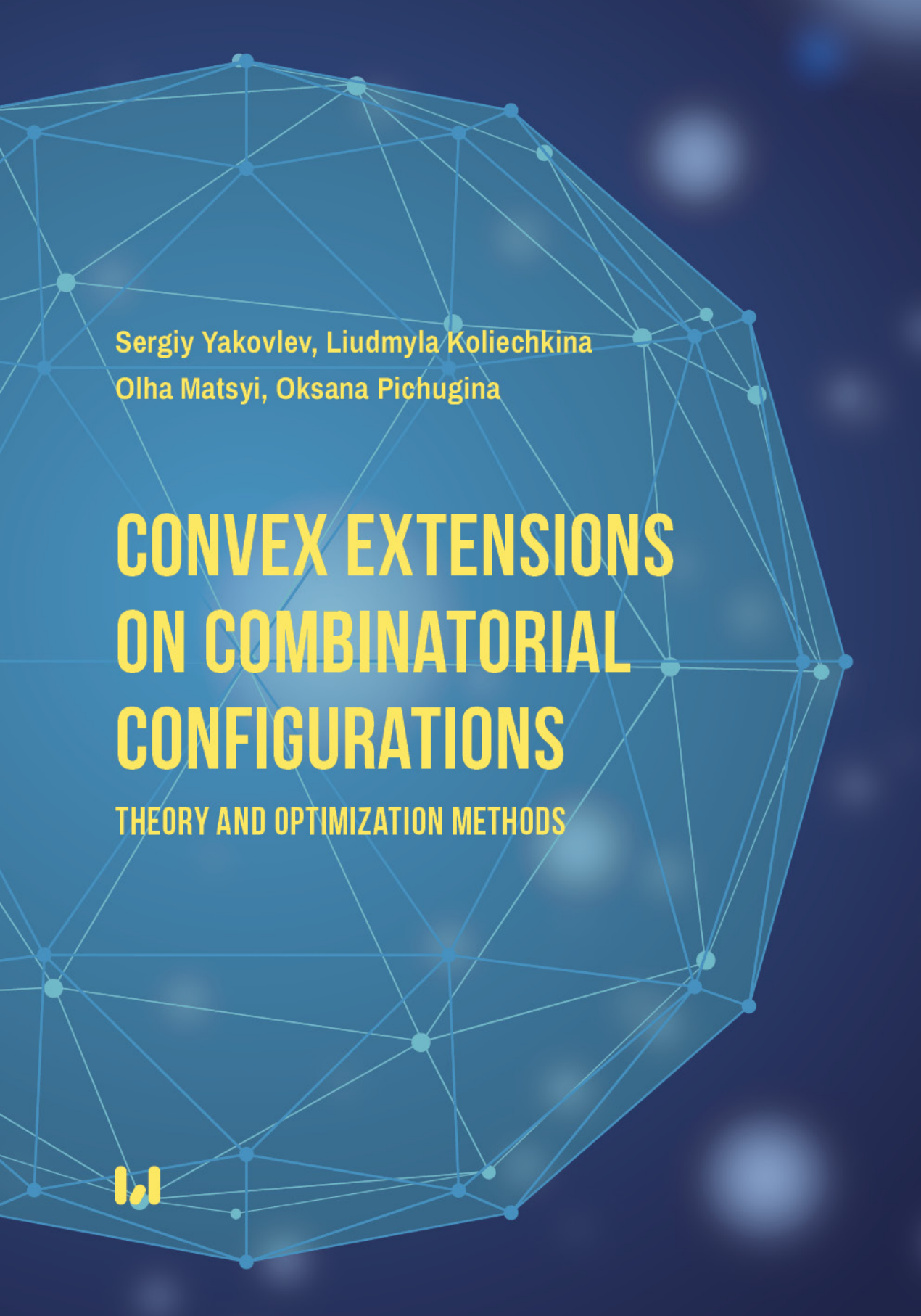




Sergiy Yakovlev, Liudmyla Koliechkina
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CONVEX EXTENSIONS ON COMBINATORIAL CONFIGURATIONS

THEORY AND OPTIMIZATION METHODS



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Introduction

Combinatorial optimization occupies a central place in discrete mathematics and operations research, providing the mathematical foundation for a broad class of problems arising in scheduling, logistics, network design, and engineering. The theoretical and algorithmic foundations of this field have been developed in the fundamental works of Papadimitriou and Steiglitz [1], Korte and Vygen [2], Schrijver [3], and Nemhauser and Wolsey [4, 5], as well as in the comprehensive handbook edited by Pardalos, Du, and Graham [6], which together form the classical canon of the discipline. Significant contributions to the development of models, methods, and the classification of combinatorial optimization problems have been made by Ukrainian researchers [7–17].

A conceptually distinct direction within combinatorial optimization arises when the elements of the feasible set are represented as points in Euclidean space. The abstract notion of a combinatorial configuration, introduced by Berge [18], provided the combinatorial foundation for this approach.

The decisive step was taken by Yu.G. Stoyan in [19, 20]: he introduced the class of Euclidean combinatorial sets, defined precisely so as to admit a natural immersion into arithmetic Euclidean space. What makes this contribution scientifically decisive is not the mapping itself, since infinitely many bijections from a combinatorial set into Euclidean space exist, but the specific class of mappings proposed by Yu. G. Stoyan, whose images possess a distinguished geometric property: they form vertex-located and polyhedral-spherical sets.

The Euclidean images of such sets are referred to in this monograph as combinatorial point configurations, and the corresponding class of nonlinear optimization problems arising from this framework is known as Euclidean combinatorial optimization [21, 22]. This structural property of the images is precisely what makes the application of convex analysis possible and underpins all subsequent theoretical developments. These foundational ideas proved extraordinarily fertile: they gave rise to a new and rapidly developing direction, which has been advanced by Stoyan's colleagues and followers across a broad range of theoretical and applied problems. A comprehensive account of this development, covering the current state and prospects of the field, is given in [22].

Within this research program, the theoretical foundations were laid in [23–25], where the theory of convex extensions of functions defined on vertices of convex polyhedra was established, the first estimates of extrema of convex functions on combinatorial polyhedra and their convex supersets were obtained, and these results were systematized within the broader framework of combinatorial optimization. The subsequent development of the geometric and analytical properties of combinatorial point configurations, including their convex hulls, vertex structure, functional representations, and polyhedral-spherical geometry, was carried out in [26–30]. These results, together with a systematic treatment of combinatorial point configurations and their associated polytopes, were consolidated in the monograph [31], which provides the geometric foundation upon which the present work is built.

Building on this foundation, the monograph develops optimization methods that exploit the specific geometric structure of combinatorial configurations after their Euclidean immersion. Polyhedral and spherical relaxations, branch-and-bound schemes adapted to the combinatorial geometry of the feasible set, penalty function methods, and cutting-plane approaches are treated in Chapter 5, not as general-purpose algorithms, but as methods whose efficiency derives directly from the properties established in the preceding chapters. This connection between the theoretical and algorithmic parts is what distinguishes the present work from existing treatments of combinatorial optimization.

The monograph is organized as follows.

Chapter 1 establishes the geometric and analytical foundations necessary for all subsequent constructions. Finite point configurations embedded in Euclidean space are formalized, and their metric, affine, and convex structures are studied. The central notion of a vertex-located configuration, one that coincides with the vertex set of its convex hull, is introduced and characterized. This notion, which may appear technical at first glance, turns out to be the decisive structural condition for the entire extension theory.

Decompositions of configurations induced by families of parallel hyperplanes are developed, revealing directional stratifications of the configuration and connecting its layered structure to the facial geometry of the convex hull. These decompositions are then generalized to decompositions generated by strictly convex functions and their level surfaces, leading to the classes of spherically located and polyhedral-spherical configurations.

A key result establishes that every finite spherically located configuration is automatically vertex-located, providing a bridge between smooth and polyhedral convex geometry. The chapter concludes with the theory of functional representations, in which configurations are encoded through systems of functional inequalities. This analytic apparatus directly prepares the ground for the extension constructions of the following chapter.

Chapter 2 undertakes a methodological reversal relative to Chapter 1. Whereas Chapter 1 employed functions as analytic instruments for encoding geometric prop-

erties of point sets, here the roles are exchanged: point configurations serve as extremal supports, and functions defined on them become the primary objects of study. The central question is the following: given an arbitrary function prescribed on a finite vertex-located configuration, can it be extended to the convex superset in such a way that convexity is ensured, and if so, how? The extremal geometry of vertex-located configurations plays a decisive role: since no configuration point is a convex combination of the others, convexity imposes no internal compatibility constraints on the prescribed values, and extensions exist for arbitrary functional data.

The chapter develops this theory in three stages. First, the extension problem is formulated precisely and a hierarchy of extension classes is introduced, from general convex extensions to strongly convex and differentiable ones, each defined by progressively stronger regularity conditions. Second, constructive existence theorems are established for each class: the minimal convex extension is built via extremal interpolation over vertex decompositions of the carrier polyhedron, strongly convex extensions through quadratic regularization, and differentiable extensions through infimal convolution with smooth kernels. Third, the extremal theory is developed: lower bounds for the minimum of a function over the configuration are derived from properties of its convex extension, yielding a hierarchy of subgradient, gradient, and quadratic estimates together with sufficient optimality conditions.

The chapter closes with the extension of the theory to functions defined on hyperspheres, treating the spherically located configurations. The results of this chapter are purely abstract, concerning arbitrary finite point configurations without any combinatorial structure. This generality is intentional: the theory is developed once, in its entirety, and then applied to specific combinatorial point configurations.

Chapter 3 introduces the specific combinatorial point configurations to which the theory of Chapters 1 and 2 is applied. The chapter focuses exclusively on ordered combinatorial objects, specifically permutations and arrangements with and without repetitions, together with their partial and Boolean variants. This restriction is motivated by three structural properties that make such objects particularly suitable for geometric realization: each object is naturally represented as a fixed-length sequence, positional information is intrinsic to its structure, and the length of the sequence is known in advance. These properties allow ordered combinatorial objects to be interpreted directly as vectors in a finite-dimensional Euclidean space.

The chapter proceeds from the most structured to the most general. Full multipermutation configurations, generated by all coordinate permutations of a fixed multiset, are studied first, and their convex hulls, the multipermutohedra, are characterized in detail. Partial multipermutation configurations, partial permutation configurations, and Boolean sets are then introduced and analyzed. The unifying theme of the entire chapter is the identification of a class of vertex-located configurations, which allows for the direct application of the theoretical results from Chapter 2.

Chapter 4 brings together the vertex-located combinatorial point configurations and the abstract extension theory. Its goal is to make the theory concrete and computationally explicit for each of the principal configuration classes. The vertex-located property guarantees applicability, while the specific structure of multipermutohedra allows explicit construction of convex extensions for natural classes of functions. Separable functions, linear functions, and distance-type functions receive detailed treatment, and for each class the resulting extremal estimates take explicit closed forms expressible through simple ordering operations on the function coefficients. The analysis is then extended to general convex functions on multipermutation configurations, and subsequently to partial permutation configurations, partial multipermutation configurations, and Boolean sets. The chapter thus transforms the existence results of Chapter 2 into explicit analytical tools: given a specific combinatorial configuration and a specific function class, one obtains not merely the guarantee that a convex extension exists, but its concrete form and the resulting computable bounds on the minimum.

Chapter 5 develops optimization methods for problems on vertex-located and polyhedral-spherical combinatorial point configurations, exploiting the geometric and analytical structure established in the preceding chapters. The original combinatorial optimization problem is reformulated as a nonlinear programming problem via Euclidean immersion, and the vertex-located property of the configuration ensures that this reformulation is exact.

Two types of relaxation are developed and compared: polyhedral relaxation, which replaces the combinatorial configuration by its convex hull, and spherical relaxation, which replaces it by the circumscribed hypersphere, yielding a simpler feasible domain at the cost of a weaker bound. These relaxations underlie a branch-and-bound scheme adapted to the combinatorial geometry of the feasible set, where the branching strategy exploits decompositions of combinatorial configurations into vertex-located subsets and the bounding strategy uses the convex extension estimates. A penalty function method for optimization on polyhedral-spherical configurations is then developed, where their structure allows the feasibility constraint to be penalized analytically, transforming the constrained problem into an unconstrained one amenable to gradient methods.

The analytical backbone of this monograph draws on several classical sources. The theory of convex functions, subdifferentials, and duality is developed in [32]. The geometry of convex polytopes, including face lattices, vertex characterizations, and combinatorial structure, is treated in [33 - 35]. The combinatorial geometry of point configurations and polytopes, including their classification and applications, is studied in [36, 37]. The results presented in Chapters 2 through 5 of this monograph were developed over a series of publications by the authors and their collaborators. Readers seeking extended proofs, detailed constructions, and supplementary material are referred to these works, cited below according to their thematic contribution.

The theory of functional representations of finite point configurations, including continuous and analytic descriptions of combinatorial feasible domains, is developed in [38–44]. Convex extensions of functions on vertex-located configurations, covering existence theorems, constructive schemes, and structural properties, are treated in [23–25, 38–48]. The extension theory for functions defined on hyperspheres is developed in [49].

The geometry and combinatorial structure of specific configuration classes, including permutation and multipermutation configurations, partial permutations, Boolean sets, and their associated polytopes, are investigated in [26, 28, 37, 42, 44]. Quadratic optimization on polyhedral-spherical sets, including the first treatment of spherical relaxation for quadratic functions, is developed in [45]. Optimization methods on combinatorial configurations, including polyhedral and spherical relaxations are developed in [45–49].

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Chapter 1

Geometric and Analytical Foundations of Finite Point Configurations

Optimization and continuation problems on combinatorial structures require analytical frameworks capable of bridging discrete configuration geometry with continuous convex analysis. Finite point sets, which serve as primary carriers of combinatorial information, possess intrinsic geometric structure that can be systematically exploited to construct such bridges. This chapter establishes the geometric and analytic foundations on which the extension and optimization theory of subsequent chapters rests.

Decompositions of configurations induced by families of parallel hyperplanes are then developed, revealing layered directional structure and stratified carrier geometry (§1.2). Hyperplane decompositions allow a configuration to be viewed not as an unstructured finite set but as an ordered geometric system, stratified according to the values of linear functionals. This ordering structure proves essential for the subsequent construction of functional representations and convex extensions. Convex-surface decompositions follow, introducing curvature-governed global carriers and polyhedral spherical hybrid embeddings (§1.3). Within such embeddings, configurations acquire smooth convex boundary placement, and the interplay between polyhedral and spherical geometry gives rise to the combined carrier structures that play a central role in the optimization methods of Chapter 5.

The chapter concludes with functional representation systems, in which configurations are encoded as intersection carriers of sublevel sets of analytic functions (§1.4). Exact and admissible representations are distinguished, minimal and irreducible functional carriers are constructed, and equivalence relations between analytic encodings are established. Through convex and quasiconvex functional systems, convex hulls and surface carriers acquire analytic reconstruction within inequality frameworks.

Together, these constructions provide a unified geometric and analytic language for describing finite point configurations. The concepts and results developed here, namely vertex-located configurations, carrier polytopes, decomposition structures, and functional representations, form the structural basis for the convex extension theory of Chapter 2 and the combinatorial configuration hierarchy of Chapter 3.

§1.1. Finite Point Configurations and Associated Polytopes

In this monograph we study finite collections of points endowed with geometric structure and interpreted not merely as discrete sets, but as integral geometric objects. Such sets naturally arise in problems of geometry, discrete analysis, and optimization, where feasible domains are represented by finite sets of points, and the properties of these domains are determined not by individual elements, but by their mutual spatial arrangement and the geometric relations induced by their embedding.

We begin with the most general notion underlying all subsequent constructions.

Definition 1.1. A finite set $E = \{x^1, x^2, \dots, x^N\}$, $N \geq 2$, consisting of pairwise distinct elements is called a finite point set.

The restriction $N \geq 2$ excludes the degenerate case of a singleton, which does not generate nontrivial affine or convex relations and therefore cannot serve as a meaningful carrier of geometric structure.

At this level the elements x^i are regarded as abstract points. In order to introduce geometric analysis, we consider embeddings of finite point sets into Euclidean space $\mathbb{R}^n$. Suppose that each element x^i is associated with a vector

$$x^i = (x_1^i, x_2^i, \dots, x_n^i) \in \mathbb{R}^n, \quad i = 1, \dots, N.$$

This embedding equips the finite set with metric, affine, and convex structures, transforming it into an object suitable for geometric investigation.

Definition 1.2. A finite point set embedded in $\mathbb{R}^n$ will be called a finite point configuration.

Thus, throughout this monograph, the term *point configuration* refers to a finite subset of $\mathbb{R}^n$ considered together with the geometric relations induced by the surrounding space. The ordering of points in the representation of E plays no role; the point configuration is treated as an unordered set.

Point configurations inherit several mutually interacting structures. They possess a metric structure through distances between points, an affine structure through affine combinations, a convex structure through convex hulls, and a combinatorial structure through the incidence relations of the faces of the convex hull. It is precisely the combination of these structures that makes finite point configurations convenient and universal objects for geometric and analytical investigations.

Every point configuration $E \subset \mathbb{R}^n$ naturally determines its affine hull, defined as the set of all affine combinations of its points,

$$\text{aff } E = \left\{ \sum_{i=1}^N \lambda_i x^i \mid \sum_{i=1}^N \lambda_i = 1 \right\}.$$

The affine hull is the minimal affine subspace containing all points of the configuration.

Definition 1.3. The dimension of a point configuration is defined as the dimension of its affine hull and is denoted by $\dim E = \dim(\text{aff } E)$.

If $\dim E = n$, the configuration is called full-dimensional; otherwise it lies in a proper affine subspace of $\mathbb{R}^n$. In many theoretical considerations one assumes, without loss of generality, that the configuration is given in a space of minimal dimension. If this is not the case, one can pass to an affinely equivalent configuration embedded in a lower-dimensional space.

A central geometric object associated with any point configuration is its convex hull,

$$\text{conv } E = \left\{ \sum_{i=1}^N \lambda_i x^i \mid \lambda_i \geq 0, \sum_{i=1}^N \lambda_i = 1 \right\}.$$

The convex hull of a finite set is a polytope. Thus every configuration determines a uniquely defined convex polytope whose geometric and combinatorial properties reflect the structure of the underlying discrete set. The convex hull may therefore be interpreted as a continuous geometric envelope of the configuration, accumulating its affine relations into a compact convex body.

The dimension of $\text{conv } E$ coincides with $\dim E$. In particular, full-dimensional configurations generate full-dimensional polytopes in $\mathbb{R}^n$. From this viewpoint, the polytope $\text{conv } E$ serves as a geometric image of the configuration, providing a bridge between discrete point sets and continuous convex bodies.

The passage from configurations to convex polytopes leads naturally to the notion of extremality.

Let $C \subset \mathbb{R}^n$ be a convex set. A point $x \in C$ is called extremal if it cannot be represented as a nontrivial convex combination of two distinct points of C .

In the case of finite configurations extremal points of $\text{conv } E$ coincide with vertices of the associated polytope. We denote their set by $\text{vert } \text{conv } E$.

While $\text{vert } \text{conv } E \subseteq E$, equality does not hold in general: some points of the configuration may lie inside the polytope or on its lower-dimensional faces.

This distinction is fundamental. The original configuration may contain redundant points that do not influence the convex geometry of the set. From the standpoint of convex analysis, the polytope is completely determined by its vertices, and all non-extremal points may be removed without altering the convex hull.

The convex hull $P = \text{conv } E$ admits two complementary geometric descriptions illuminating different aspects of configuration structure. On the one hand, the polytope can be described directly as the convex hull of its generating points. This description reflects the discrete origin of the polytope and is commonly referred to as its vertex representation. It is particularly natural when the configuration itself is taken as primary data.

On the other hand, the same polytope can be represented as the intersection of finitely many closed half-spaces,

$$P = \{x \in \mathbb{R}^n: \langle a_i, x \rangle \leq b_i\}.$$

This form is known as the hyperplane representation. Each inequality defines a hyperplane

$$H_i = \{x \in \mathbb{R}^n: \langle a_i, x \rangle = b_i\},$$

and whenever such a hyperplane is supporting, the intersection $F_i = P \cap H_i$ forms a facet of the polytope. A minimal hyperplane representation corresponds to the system of all facets.

The two representations are equivalent but emphasize different structural features. The vertex representation highlights the discrete combinatorial nature of the configuration, whereas the hyperplane representation provides an analytic description suitable for convex analysis and optimization. The normals a_i to supporting hyperplanes determine characteristic geometric directions that will later induce decompositions of point configurations.

Supporting hyperplanes generate a natural stratification of the polytope and, consequently, of the underlying configuration. Subsets of points lying on common supporting hyperplanes correspond to faces of the polytope. These faces form a hierarchical structure consisting of vertices, edges, higher-dimensional faces, facets, and the polytope itself.

This hierarchy encodes the combinatorial organization of the configuration and provides a framework for structural analysis. In particular, the incidence relations between vertices and facets determine adjacency structures that will play an important role in subsequent decompositions and functional constructions.

The geometric structures described above establish a dual perspective on finite configurations. On the one hand, configurations are discrete sets of points; on the other, they generate convex bodies whose facial and hyperplane structures admit continuous analytic description. This duality underlies the transition from discrete geometry to convex analysis that will be developed throughout the monograph.

The structural analysis of convex hulls leads naturally to the identification of a special class of configurations in which every point is geometrically essential. As noted above, an arbitrary configuration may contain points that do not influence the convex hull and therefore do not contribute to the extremal structure of the associated polytope. From both geometric and analytical perspectives it becomes natural to distinguish configurations free of such redundancy.

Definition 1.4. A point configuration $E \subset \mathbb{R}^n$ is called vertex-located if

$$E = \text{vert conv } E,$$

that is, every point of the configuration is a vertex of its convex hull.

Vertex-located configuration (VLC) represents structurally minimal carriers of convex geometry. VLCs contain neither interior points nor points lying on faces of lower dimension. Each element of the VLC directly determines the shape of the polytope and participates in the formation of its supporting hyperplanes.

This class of configurations plays a central role throughout the monograph. The importance of vertex-located configurations manifests itself on several interconnected levels.

From a purely geometric standpoint, such configurations eliminate redundancy. If a configuration is not vertex-located, then there exists at least one point $x \in E$ representable as a convex combination of other points:

$$x = \sum_{i=1}^N \lambda_i x^i, \lambda_i > 0, \sum_{i=1}^N \lambda_i = 1, x^i \neq x.$$

Such points may be removed without altering the convex hull. Consequently, every configuration induces a unique vertex-located subconfiguration consisting precisely of the vertices of $\text{conv } E$. This reduction establishes a canonical minimal representation of the convex geometry of the configuration.

From the viewpoint of polyhedral theory, vertex-located configurations correspond exactly to vertex sets of polytopes. In this sense, they form the discrete skeleton of convex bodies. All facial structures, supporting hyperplanes, and combinatorial incidences of the polytope are determined solely by these extremal points.

The analytical significance of VLCs becomes particularly evident when considering functions defined on finite sets.

Let $f: E \rightarrow \mathbb{R}$ be a real-valued function defined on a configuration. Many extremal properties of such functions, when extended to $\text{conv } E$, depend only on their values at vertices. For convex extensions, interior points impose no independent constraints: their values must be consistent with convex combinations of vertex values. Thus the admissible convex continuations of f are determined entirely by its restriction to $\text{vert conv } E$.

This observation has far-reaching consequences for the theory of continuous extensions developed in subsequent chapters. VLCs serve as natural discrete domains on which functions are specified prior to convex continuation. Any redundant interior points would introduce artificial constraints incompatible with minimal convex envelopes.

From an optimization perspective, the relevance of VLCs is equally fundamental. Many discrete optimization problems can be reformulated as optimization problems over convex polytopes obtained as convex hulls of feasible discrete sets. In such relaxations, optimal solutions of linear or convex objective functions are attained at vertices of the feasible polytope. Hence VLCs represent precisely the set of extremal feasible solutions. Studying functions and extensions on these configurations establishes a bridge between combinatorial optimization and continuous convex programming.

The reduction of arbitrary configurations to vertex-located ones therefore performs several simultaneous roles: it removes geometric redundancy, isolates extremal structure, simplifies analytic continuation, and aligns discrete optimization problems with convex relaxations.

The structural properties of configurations and their polytopes are preserved under affine transformations. Let

$$T(x) = Ax + b,$$

where A is a nonsingular matrix and $b \in \mathbb{R}^n$.

Such a mapping establishes a bijection between configurations E and $T(E)$, preserving affine relations, convex hulls, facial structures, and vertex-locatedness. Configurations related in this way are said to be affinely equivalent.

Affine equivalence plays an important methodological role. It allows one to normalize configurations without altering essential geometry. For example, configurations may be translated so that their centroid coincides with the origin, rescaled to simplify analytic expressions, or transformed to reduce degeneracy. These operations simplify subsequent constructions while preserving combinatorial and convex properties.

Not all configurations are full-dimensional. Degenerate situations arise when points lie in lower-dimensional affine subspaces. For instance, a configuration in $\mathbb{R}^n$ may lie entirely within a hyperplane or even a lower-dimensional flat. Such configurations are referred to as degenerate or partially degenerate.

Degeneracy does not eliminate geometric interest. On the contrary, many naturally arising configurations possess intrinsic lower-dimensional structure. In such cases it is convenient to pass to an affinely equivalent configuration embedded in a space of minimal dimension. This reduction simplifies analytic representations and avoids artificial singularities in subsequent constructions.

The interplay between configurations and their convex hulls reveals a deep bidirectional relationship. On the one hand, the arrangement of points determines the geometry of the polytope: its dimension, number of facets, adjacency relations, and combinatorial type. On the other hand, the structure of the polytope constrains the possible configurations that can generate it. Thus discrete and continuous structures are mutually interdependent.

This duality is one of the central themes of the present monograph. Finite point configurations, though discrete, generate convex bodies whose analytic and geometric properties can be studied using the apparatus of convex analysis. Conversely, convex bodies may be analyzed through the combinatorial structure of their vertex configurations.

Finite configurations therefore occupy an intermediate position between discrete and continuous geometric worlds. They serve simultaneously as combinatorial objects and as generators of convex continua. This dual nature makes them

particularly suitable for constructing transitions between discrete optimization problems and continuous analytical models.

The geometric framework established in this section forms the foundation for subsequent developments. In the following section, point configurations will be studied through decompositions induced by families of hyperplanes. Such decompositions reveal layered structures within configurations and provide the first step toward functional and surface representations. These constructions, in turn, prepare the ground for the theory of continuous convex extensions that constitutes one of the central themes of this monograph.

§1.2. Decompositions of Point Configurations and Their Convex Hulls

In the previous section it was shown that a finite point configuration $E \subset \mathbb{R}^n$ may be regarded as a discrete carrier of a convex polytope $P = \text{conv } E$. However, such a global characterization often proves insufficient for a detailed analysis of the internal structure of the configuration. In order to reveal finer geometric properties, it is natural to consider decompositions of the configuration into subsets determined by families of hyperplanes or surfaces.

In the most general form, by a decomposition of a point configuration we shall understand a representation

$$E = \bigcup_k E_k, \quad E_k \cap E_l = \emptyset, \quad k \neq l,$$

where the subsets E_k are defined by some geometric rule applied uniformly to all points of the configuration.

In the present section, primary attention is devoted to decompositions induced by:

- linear functions and families of parallel hyperplanes;
- convex functions and corresponding level surfaces.

Such decompositions allow one to consider a point configuration not as an unstructured finite set but as an ordered geometric system, which is essential for subsequent analysis.

We begin with decompositions generated by linear functions. Consider a linear function

$$\ell(x) = \langle a, x \rangle, \quad a \in \mathbb{R}^n, \quad a \neq 0.$$

For a fixed vector a , consider the family of parallel hyperplanes

$$H_\alpha = \{x \in \mathbb{R}^n : \langle a, x \rangle = \alpha\}, \quad \alpha \in \mathbb{R}.$$

Since the configuration E is finite, the restriction of $\ell(x)$ to E produces a finite set of values $\alpha_1 < \dots < \alpha_m$.

Accordingly, the configuration E is decomposed into subsets

$$E_k = \{x \in E: \langle a, x \rangle = \alpha_k\}, k = 1, \dots, m.$$

This representation will be called a decomposition of the configuration by parallel hyperplanes in direction a . The subsets E_k are referred to as the levels of the configuration with respect to the direction a .

Geometrically, this decomposition corresponds to scanning the configuration by a family of parallel hyperplanes orthogonal to a . As the parameter α varies, the hyperplanes sweep through the configuration and reveal its layered structure. This interpretation is particularly useful when the configuration is associated with a convex polytope.

Let $P = \text{conv } E$ be the convex hull of the configuration. The extreme level

$$E_m = \left\{ x \in E : \langle a, x \rangle = \max_{y \in E} \langle a, y \rangle \right\}$$

lies on the hyperplane

$$H = \{x \in \mathbb{R}^n: \langle a, x \rangle = \beta\},$$

Where $\beta = \max_{y \in E} \langle a, y \rangle$.

This hyperplane is supporting for the polytope P . If the direction a coincides with the normal to a facet of P , the level E_m coincides with the set of vertices of that facet. Thus extreme levels correspond to supporting hyperplanes and reflect the facial structure of the convex hull.

Internal levels E_k , $1 < k < m$, contain points that are not extremal in the direction a . Such points may lie in the interior of the polytope or belong to faces of lower dimension.

The number of levels mmm generated by the configuration in a given direction is an important structural characteristic. A configuration is called m -level in direction a if the decomposition induced by hyperplanes orthogonal to a contains exactly mmm nonempty levels.

Configurations with small numbers of levels are of particular interest. For example, two-level configurations consist of points lying on two parallel hyperplanes. More generally, configurations with few levels possess a simple layered structure that strongly influences the behavior of functions defined on E and the possibilities for their continuation and optimization.

The number of levels may also be interpreted as a measure of geometric complexity. Configurations generating few levels exhibit pronounced geometric regularity, whereas configurations producing many levels reflect more intricate spatial distributions of points.

This characteristic interacts with affine dimension. Configurations contained in lower-dimensional affine subspaces necessarily generate fewer levels than full-dimensional configurations under generic directions.

An important property of level decompositions is their stability with respect to small perturbations of the direction vector.

Let a' be close to a . For sufficiently small perturbations, the ordering of the values $\langle a', x \rangle$ on the finite set E remains unchanged, and the induced level decomposition coincides with the original one.

This stability admits a natural geometric interpretation. Each facet F of the polytope $P = \text{conv } E$, generates a normal cone consisting of all directions for which F remains an extreme face with respect to the corresponding linear functional.

Inside the interior of such a cone the ordering of the functional values on E is preserved, and the level decomposition remains unchanged.

Consequently, the space of directions can be partitioned into finitely many regions, each corresponding to a fixed combinatorial structure of the decomposition.

From this viewpoint, level decompositions form continuous families parameterized by regions of the normal fan of the polytope. Each region of the fan corresponds to a class of equivalent decompositions possessing identical layered structure.

Different directions therefore produce different stratifications of the same configuration. If the direction is orthogonal to a facet, the extreme level reveals the structure of that facet. For generic directions the extreme levels may consist of single points corresponding to isolated vertices of the convex hull. If the direction reflects symmetry of the configuration, several geometrically equivalent points may lie on the same level.

Thus the family of all level decompositions contains information both about the convex hull and about the internal geometric organization of the configuration.

Hyperplane decompositions represent only a linear instance of a more general geometric construction. If the linear function $\ell(x)$ is replaced by a strictly convex function $f(x)$, the family of parallel hyperplanes is replaced by a family of nested level surfaces

$$S_\alpha = \{x \in \mathbb{R}^n : f(x) = \alpha\}.$$

In this setting the principal structural properties remain the same: extreme levels correspond to extremal points relative to the function f , internal levels form a nested layered structure, and the decomposition remains finite when restricted to a finite configuration.

Such surface-based decompositions substantially extend the range of admissible geometric representations and provide a natural framework for nonlinear continuation procedures.

From an analytical viewpoint, hyperplane-induced decompositions serve as a bridge between discrete point configurations and continuous convex structures. They preserve the layered organization of the configuration while revealing the

geometry of its convex hull and preparing the transition to nonlinear geometric representations.

For this reason decompositions by parallel hyperplanes form a fundamental analytical tool in the study of combinatorial point configurations and their functional extensions.

§1.3. Decompositions of Finite Point Configurations by Convex Surfaces

In §1.2 decompositions generated by linear functions and families of parallel hyperplanes were studied. Such decompositions reveal layered structures determined by directional projections and supporting hyperplanes of convex hulls. Their geometric interpretation is closely related to the facial structure of the polytope $P = \text{conv } E$.

However, hyperplane decompositions are intrinsically directional: their structure depends on the choice of a fixed vector a . Consequently, they capture only partial geometric information about the configuration.

In many geometric and analytical problems this restriction becomes significant. In particular, when studying global continuation procedures or convex envelopes, it is desirable to employ constructions reflecting the overall convex geometry of the configuration rather than its behavior along individual directions.

This motivates the introduction of decompositions generated by convex functions.

Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous strictly convex function. For each value α consider the level surface

$$S_\alpha = \{x \in \mathbb{R}^n: f(x) = \alpha\}$$

and the corresponding sublevel set

$$C_\alpha = \{x \in \mathbb{R}^n: f(x) \leq \alpha\}.$$

By convexity of the function f , the sets C_α form a nested family of convex bodies whose boundaries are the surfaces S_α .

Restricting the function f to a finite point configuration $E \subset \mathbb{R}^n$ yields a finite ordered set of values $\alpha_1 < \dots < \alpha_m$. This induces a decomposition

$$E = \bigcup_{k=1}^m E_k, \quad E_k = \{x \in E: f(x) = \alpha_k\}.$$

Such a representation will be called a convex-surface decomposition of the configuration induced by the function f .

Unlike hyperplane decompositions, surface decompositions do not privilege any particular direction. Instead, they reflect the global geometry of the convex body generated by the sublevel sets of f . Geometrically, the decomposition may be interpreted as a gradual expansion of the convex body C_α whose boundary successively captures points of the configuration.

A particularly transparent case arises when the level surfaces are spheres.

The simplest and most geometrically transparent example of a convex-surface decomposition is obtained when level surfaces are spheres. This observation motivates the introduction of configurations whose points lie on a common spherical surface.

Definition 1.5. A finite point configuration $E \subset \mathbb{R}^n$ is called spherically located if there exist a point $\tau \in \mathbb{R}^n$ and a radius $r > 0$ such that

$$E \subset \{x \in \mathbb{R}^n : \|x - \tau\| = r\}.$$

Spherically located configurations arise naturally in the study of extremal support directions, normal structures, and symmetric decompositions. They may be viewed as configurations whose geometric carrier is not polyhedral but smooth and strictly convex.

The spherical case represents only a particular instance of a broader geometric phenomenon. Convex-surface decompositions may be generated by arbitrary strictly convex functions, whose level surfaces define smooth convex bodies. This leads to the identification of configurations lying on general convex surfaces.

Definition 1.6. A finite point configuration $E \subset \mathbb{R}^n$ is called convex surface-located if there exist a strictly convex function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ and a value α such that

$$E \subset \{x \in \mathbb{R}^n : f(x) = \alpha\}.$$

Geometrically, all points of the configuration lie on the boundary of a convex body determined by the sublevel set $\{x: f(x) \leq \alpha\}$. Strict convexity ensures that the surface contains no flat regions, and therefore each point of the configuration occupies a geometrically essential position relative to the surface.

Convex surface-located configurations form a natural nonlinear analogue of vertex-located configurations introduced earlier. While vertex-located configurations are extremal relative to polyhedral convexity, convex surface-located configurations are extremal relative to smooth convex geometry.

This parallelism is not merely formal. In continuation problems, convex extensions defined over polytopes depend on vertex data, whereas extensions defined relative to smooth convex carriers depend on surface data. Convex surface-located configurations therefore provide an intermediate geometric framework bridging discrete convexity and smooth convex analysis.

A particularly important subclass is obtained when spherical location and vertex location are simultaneously satisfied.

Definition 1.7. A finite configuration $E \subset \mathbb{R}^n$ is called polyhedral–spherical if it is both spherically located; and vertex-located, i.e.

$$E = \text{vert conv } E.$$

Polyhedral–spherical configurations combine extremality in the polyhedral sense with global coherence relative to a smooth convex surface. Such configurations are especially convenient for studying normal cones, facet structures, and continuation schemes defined over spherical carriers.

Among convex-surface decompositions, the spherical case plays a distinguished role. The sphere is the simplest strictly convex surface and therefore provides the most transparent geometric model for studying extremal properties of configurations relative to smooth convex carriers.

As will be shown below, spherical location automatically implies that all points of the configuration are vertices of the convex hull, thereby establishing a direct link between smooth and polyhedral convex geometry.

This observation motivates the introduction of configurations simultaneously possessing spherical location and polyhedral extremality. Such configurations combine polyhedral extremality with global coherence relative to a smooth convex surface and prove especially convenient for studying normal cones, facet structures, and continuation schemes defined over spherical carriers.

To clarify the structural role of spherical location, we establish the following fundamental result.

Theorem 1.1. Every finite spherically located configuration is vertex-located and hence polyhedral–spherical.

Proof. Let

$$E = \{x^1, \dots, x^N\} \subset S(\tau, r)$$

be a configuration whose points lie on a sphere of radius r centered at τ . Suppose that a point $x \in E$ is not a vertex of $\text{conv } E$. Then x admits a nontrivial convex representation

$$x = \sum_{i=1}^N \lambda_i y^i, y^i \in E, \lambda_i > 0, \sum_{i=1}^N \lambda_i = 1.$$

Introduce shifted vectors $u = x - \tau, v^i = y^i - \tau$. Since all points lie on the sphere,

$$\|u\| = \|v^i\| = r.$$

The convex representation becomes

$$u = \sum_{i=1}^N \lambda_i v^i.$$

Consider the strictly convex function

$$\varphi(z) = \|z\|^2.$$

By strict convexity,

$$\left\| \sum_{i=1}^N \lambda_i v^i \right\| < \sum_{i=1}^N \lambda_i \|v^i\|^2,$$

provided that at least two vectors v_i are distinct.

Since

$$\sum_{i=1}^N \lambda_i \|v^i\|^2 = r^2,$$

we obtain $\|u\|^2 < r^2$, which contradicts $\|u\| = r$.

Hence the assumption was false, and x must be a vertex of $\text{conv } E$. Since x was arbitrary, all points of E are vertices, i.e., $E = \text{vert conv } E$. Thus every finite spherically located configuration is vertex-located. ■

This theorem establishes a fundamental link between smooth and polyhedral convex geometry: strict convexity enforces extremality. Points lying on a strictly convex surface cannot be represented as convex combinations of other surface points without violating strict convexity. Consequently, spherical configurations represent smooth analogues of vertex-located polyhedral configurations. The sphere acts as a strictly convex carrier eliminating interior redundancy.

Remark 1.1. Although spherical location automatically implies vertex-locatedness for finite configurations, the terminology *polyhedral-spherical* remains useful. It emphasizes that the configuration simultaneously admits two complementary geometric interpretations: a polyhedral one, through the convex hull $\text{conv } E$, and a smooth one, through the spherical carrier.

The structural rigidity of polyhedral-spherical configurations manifests itself in several aspects.

First, such configurations possess a dual geometric character. They are extremal relative to their convex hull while simultaneously lying on a smooth strictly convex surface.

Second, their normal structures admit compatible interpretations. Normals to facets of $\text{conv } E$ correspond to outward normals of the spherical surface at the same points, linking polyhedral and differential-geometric viewpoints.

Third, polyhedral-spherical configurations provide particularly stable frameworks for decomposition procedures. Hyperplane decompositions reflect the polyhedral structure of the configuration, whereas surface decompositions reflect the geometry of the spherical carrier. The compatibility of these two perspectives leads to analytically convenient layered representations.

These observations clarify the relationship between the geometric classes introduced in this section. They form a hierarchy of increasing geometric rigidity:

finite configurations $\rightarrow$ *vertex-located* $\rightarrow$ *convex surface-located* $\rightarrow$
spherical $\rightarrow$ *polyhedral-spherical*.

Each successive class imposes additional convexity or extremality constraints, resulting in configurations with stronger geometric determinacy.

From the standpoint of continuation theory, convex-surface carriers play a particularly important role. When extending functions from discrete configurations to continuous domains, convex surfaces provide natural envelopes controlling admissible extensions. Hyperplane decompositions describe configurations through directional slicing, whereas surface decompositions capture their placement relative to global convex envelopes.

Together these two mechanisms form a geometric framework connecting discrete configurations with continuous convex structures.

The constructions developed in this section remain geometric in nature: configurations are described through their position relative to hyperplanes and convex surfaces. For continuation and optimization problems, however, it becomes necessary to replace geometric carriers with analytic representations.

Such a transition is achieved by representing configurations as level sets of appropriately constructed functions. In this functional framework convex hulls, surface carriers, and continuation structures admit analytic descriptions suitable for optimization and variational analysis.

This transition leads from geometric decomposition theory to functional representation theory, which will be developed in the next section.

The geometric constructions developed in this section remain purely spatial in nature: configurations are described through their placement relative to hyperplanes and convex surfaces. However, for continuation and optimization problems it becomes necessary to replace geometric carriers with analytic ones.

Such a transition is achieved by representing configurations as level sets or zero sets of specially constructed functions. In this functional framework, convex hulls, surface carriers, and continuation structures acquire analytic representations amenable to optimization and variational analysis.

This transition marks the passage from geometric decomposition theory to functional representation theory.

In the next section, continuous functional representations of point configurations will be introduced. It will be shown how configurations may be encoded through systems of convex functions, how their convex hulls arise as sublevel sets, and how such representations provide the analytical apparatus required for constructing convex continuations and solving optimization problems on combinatorial point configurations.

§1.4. Functional Representations of Point Configurations

The geometric constructions developed in the preceding sections described finite point configurations through their spatial placement relative to hyperplanes, convex polytopes, and convex surfaces. These constructions revealed layered decompositions, extremal structures, and geometric carriers associated with configurations embedded in $\mathbb{R}^n$.

In particular, vertex-located configurations were characterized through extremality with respect to convex hulls, hyperplane decompositions generated directional stratifications of configurations, and convex-surface carriers described configurations embedded in smooth strictly convex boundaries. Together these approaches provided a geometric framework for analyzing discrete point sets within convex geometry.

However, in many analytical and variational problems geometric descriptions alone are not sufficient. When configurations serve as carriers of functional data, or when discrete values assigned to configuration points must be extended to continuous domains, it becomes necessary to encode configurations through analytic objects. This leads naturally to the notion of functional representations of point configurations.

Let $E \subset \mathbb{R}^n$ be a finite configuration. In general, a functional representation is obtained by introducing a vector-valued mapping

$$F: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

together with a set $C \subset \mathbb{R}^m$ and representing the configuration through the relation

$$E = \{x \in \mathbb{R}^n : F(x) \in C\}.$$

Equivalently,

$$E = F^{-1}(C).$$

Thus the configuration appears as the preimage of a carrier set under the mapping F . The analytic and geometric properties of such a representation depend jointly on the structure of the mapping F and on the geometry of the carrier set C . Different choices of F and C generate representation systems with different convexity classes, smoothness properties, dimensional characteristics, and structural complexity.

In many situations the functional system must reproduce the configuration exactly. This leads to the notion of exact functional representations.

Definition 1.8. Let $E \subset \mathbb{R}^n$ be a finite configuration. A pair (F, C) , where $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a mapping and $C \subset \mathbb{R}^m$ is a set, is called an exact functional representation of E if $E = F^{-1}(C)$.

In this case the mapping F is said to represent the configuration exactly with respect to the carrier set C .

Exact functional representations translate geometric properties of configurations into analytic properties of function systems. For instance, convexity of the carrier set reflects convexity properties of analytic envelopes, while smoothness of the mapping F determines the regularity of analytic boundaries defining the configuration. In this sense functional representations convert discrete geometric objects into analytic carriers described through systems of functional constraints.

At the same time exact representations are generally non-unique. A given configuration may admit many analytic encodings differing in functional dimensionality, convexity class, algebraic structure, or smoothness properties. Such nonuniqueness reflects the diversity of analytic carriers compatible with a fixed discrete geometry.

In many applications, however, it is not necessary to reproduce the configuration exactly. Instead, the analytic representation must preserve only those structural properties that are relevant for the problem under consideration. This leads to the notion of admissible functional representations.

Definition 1.9. Let $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$. The representation $E \subset F^{-1}(C)$ is called admissible if the analytic carrier $F^{-1}(C)$ contains the configuration and preserves structural properties required for subsequent geometric or analytic constructions.

Admissible representations therefore provide analytic carriers compatible with the geometry of the configuration without necessarily reproducing the discrete set exactly. The distinction between exact and admissible representations determines the analytic flexibility of configuration encoding. Exact systems reproduce discrete geometry without deformation, whereas admissible systems generate analytic carriers capable of supporting additional structural constructions within convex, quasiconvex, or hybrid analytic frameworks.

Functional representations may also be interpreted as analytic counterparts of the geometric constructions studied earlier. Hyperplane decompositions correspond to systems of affine functions generating polyhedral carriers, while convex-surface decompositions correspond to level structures of strictly convex functions generating smooth carriers. In this way functional representation theory extends geometric carrier analysis into the analytic domain, replacing spatial stratifications with systems of functional relations.

Consider a mapping

$$F(x) = (f_1(x), f_2(x), \dots, f_m(x)),$$

where $f_i: \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous functions.

The system of inequalities

$$f_i(x) \leq 0, \quad i = 1, \dots, m.$$

generates an analytic carrier of the form

$$D(F) = \bigcap_{i=1}^m \{x \in \mathbb{R}^n : f_i(x) \leq 0\}.$$

If $E = D(F)$, then the configuration is represented exactly as the intersection of sublevel sets induced by the functions f_i .

Such intersection representations generalize the classical description of convex polytopes as intersections of half-spaces. In linear settings the functions f_i are affine and define polyhedral carriers. In nonlinear settings convex or quasiconvex functions generate curved analytic carriers whose boundaries reflect smooth convex geometry.

Functional intersection systems therefore provide a flexible analytic language capable of encoding both polyhedral and smooth convex carriers associated with point configurations. They allow discrete configurations to be represented through systems of functional inequalities whose structural properties reflect convexity, extremality, and carrier geometry of the underlying set.

At the same time the flexibility inherent in functional representations gives rise to structural questions concerning the defining systems of functions. When a configuration is represented as an intersection of sublevel domains, it becomes important to determine whether all defining functions contribute essentially to the representation or whether some may be removed without altering the encoded set.

The study of minimality and interdependence of defining functions therefore becomes a central aspect of the theory of functional representations and will be addressed in the subsequent sections.

This leads to the study of redundancy and minimality in functional representation systems.

Let $F(x) = \{f_1(x), \dots, f_m(x)\}$ be a functional system representing a configuration $E \subset \mathbb{R}^n$ through the intersection carrier

$$E = \bigcap_{i=1}^m \{x \in \mathbb{R}^n : f_i(x) \leq 0\}.$$

Each function f_i defines an analytic constraint restricting the admissible carrier domain. Taken together, these constraints determine the geometry of the represented configuration.

However, not all constraints need be structurally necessary. It may occur that removal of one of the defining functions does not alter the represented set. In such cases the function is redundant and does not contribute essentially to the analytic encoding of the configuration.

The identification and elimination of redundant constraints lead to the notion of minimal functional systems.

Definition 1.10. A functional system $F = \{f_1, \dots, f_m\}$ representing a point configuration E is called minimal if removal of any function f_i changes the represented set, that is,

$$E \neq \bigcap_{\substack{j=1 \\ j \neq i}}^m \{x \in \mathbb{R}^n : f_j(x) \leq 0\}.$$

for every $i = 1, \dots, m$.

Minimal systems play a role analogous to minimal supporting systems in polyhedral geometry. Just as a convex polytope may be represented as the intersection of finitely many half-spaces, some of which may be redundant, analytic carriers may admit multiple descriptions differing in structural economy. Minimal functional systems therefore correspond to irreducible analytic encodings of configuration carriers.

The structural importance of minimality becomes apparent when examining the geometry of analytic carriers. Each defining function determines a boundary component of the carrier domain. Removal of a function enlarges the domain and alters its geometric structure. Thus minimal systems correspond to carriers whose analytic boundaries are defined by irreducible functional constraints.

In vertex-located configurations, minimal functional systems often correspond to supporting function systems associated with facets of the convex hull. Each function encodes a supporting constraint, and together they reconstruct the polyhedral carrier.

In convex surface-located configurations, minimal systems arise from strictly convex carrier functions whose level surfaces pass through configuration points. In this setting irreducibility reflects curvature-induced extremality rather than facet combinatorics.

Beyond minimality, one may also consider functional interdependence within representation systems. Even when functions are not redundant in the strict sense, they may exhibit structural dependence through analytic relations or monotone transformations.

This motivates the concept of structural independence.

Definition 1.11. A functional system is structurally independent if none of its defining inequalities is implied by the others within the admissible analytic class.

Structural independence strengthens minimality by excluding analytic derivability among constraints. Independent functional systems generate carriers whose boundary components correspond to analytically distinct extremal conditions. In contrast, dependent systems encode carriers with correlated boundary structures.

Functional systems may also admit hierarchical organization. Subsystems of defining functions generate intermediate carriers containing the configuration. These nested analytic domains form stratified representation structures reflecting progressive constraint activation. Such hierarchical carriers generalize the layered geometric decompositions studied earlier into the analytic domain.

The multiplicity structure of functional systems therefore reflects the internal geometry of configuration carriers. Redundant constraints correspond to inactive boundary components, minimal systems correspond to irreducible carriers, and independent systems correspond to analytically distinct boundary structures.

From a representational standpoint, the transition from arbitrary functional systems to minimal irreducible systems parallels the transition from arbitrary hyperplane families to facet-defining systems in convex polytope theory. This parallelism further reinforces the structural correspondence between geometric and analytic representations of point configurations.

Functional minimality therefore serves as a foundational organizing principle within representation theory. By eliminating redundancy and isolating irreducible analytic constraints, one obtains carrier systems optimally adapted to the extremal structure of the configuration.

The minimal and structurally independent functional systems considered above generate analytic carriers whose geometric properties depend essentially on the convexity class of the defining functions. Among the wide spectrum of admissible functional representations, particular significance is attached to systems constructed from convex and quasiconvex functions, since their sublevel domains inherit convexity and therefore remain compatible with the convex geometric structure of the underlying configuration.

Let

$$F(x) = (f_1(x), \dots, f_m(x))$$

be a minimal functional system representing a configuration E through the intersection carrier

$$E = \bigcap_{i=1}^m \{x \in \mathbb{R}^n : f_i(x) \leq 0\}.$$

If each function f_i is convex, then every sublevel set

$$\{x \in \mathbb{R}^n : f_i(x) \leq 0\}$$

is convex, and therefore the intersection domain is also convex. In this case the functional carrier constitutes a convex analytic domain encoding the configuration through systems of convex inequalities.

Convex functional systems therefore provide analytic realizations of convex carriers associated with point configurations. Their defining inequalities play roles analogous to supporting hyperplanes in polyhedral geometry, while their sublevel boundaries generalize facet structures to nonlinear convex settings.

In vertex-located configurations, minimal convex functional systems frequently reconstruct the convex hull itself. Each defining function corresponds to a supporting constraint activated on a subset of configuration points, and together these functions generate the analytic envelope coinciding with $\text{conv } E$.

Beyond convex systems, quasiconvex functional representations extend the analytic encoding framework. A function f is quasiconvex if all its sublevel sets $\{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ are convex, even if f itself is not convex in the classical sense.

Quasiconvex systems therefore generate convex carriers while allowing broader analytic flexibility than strictly convex systems. Although quasiconvex functions may exhibit nondifferentiability or flat regions, their carriers preserve convexity of representation domains.

Functional representation systems also admit equivalence transformations. Distinct functional systems may generate identical analytic carriers, thereby encoding the same configuration through different analytic mechanisms.

Definition 1.12. Two functional systems F and G are said to be equivalent representations of a configuration E if their induced carriers coincide, that is,

$$\bigcap_i \{x \in \mathbb{R}^n : f_i(x) \leq 0\} = \bigcap_j \{x \in \mathbb{R}^n : g_j(x) \leq 0\}.$$

Equivalent systems may differ in analytic complexity, smoothness, convexity class, or dimensionality, yet encode identical carrier geometry.

Representation equivalence plays an important methodological role. It allows analytically complicated systems to be replaced by simpler or structurally minimal ones without altering the represented configuration.

Equivalence transformations may also reveal analytic redundancy beyond simple constraint elimination. Entire functional systems may sometimes be replaced by alternative systems possessing improved convexity, regularity, or minimality properties. This flexibility permits analytic normalization of representation carriers.

An important structural problem concerns the reconstruction of convex hulls through functional systems. Suppose a configuration is represented by a convex functional system satisfying appropriate minimality conditions. In such situations the resulting analytic carrier may coincide with $\text{conv } E$. Thus convex hulls can, in principle, be recovered analytically through systems of convex inequalities.

Hull reconstruction through functional systems reveals a close correspondence between extremal points and boundary activation of analytic constraints. Supporting inequalities become active precisely at extremal configuration points, thereby reproducing the supporting geometry of the convex hull.

This analytic perspective shows that convex hull structure can be described not only through geometric constructions but also through systems of functional inequalities defining suitable carrier domains. Such representations provide analytic access to extremal structure, supporting directions, and carrier boundaries.

The analytic framework developed here therefore unifies polyhedral carriers, smooth convex surfaces, and hybrid analytic envelopes within a common representation scheme based on functional inequalities.

Through convex and quasiconvex representation systems, discrete point configurations acquire analytic carriers whose geometry reflects the boundary organization and extremal properties of the underlying set. These constructions complete the representation-theoretic apparatus required for encoding configuration geometry within analytic carrier systems.

The convex and quasiconvex functional systems considered above generate analytic carriers whose structural properties depend not only on convexity class but also on geometric regularity, boundary structure, and analytic stratification of the defining constraints. Such distinctions allow representation systems to be classified according to the geometric nature of their induced carriers.

In polyhedral functional systems, carriers are generated by finite families of affine functions. Their boundaries consist of flat analytic components corresponding to supporting hyperplanes of the convex hull. Extremal configuration points activate boundary inequalities in a piecewise-linear manner, producing carriers with combinatorial facet structure. These systems therefore reproduce analytic analogues of polyhedral geometry within functional representation theory.

In contrast, strictly convex functional systems generate carriers bounded by smooth convex surfaces. Here configuration points lie on curved analytic boundaries defined by level sets of strictly convex functions. Boundary activation is governed not by facet combinatorics but by curvature-induced extremality. Such carriers provide analytic realizations of convex surface-located configurations introduced earlier.

Between these two extremes lie hybrid representation systems combining affine and nonlinear convex functions. Their carriers possess mixed boundary structure: flat components coexist with curved analytic surfaces. Hybrid carriers arise naturally when configurations exhibit both polyhedral and smooth convex geometric features. This hybrid analytic class provides representational flexibility accommodating diverse carrier geometries.

Representation systems may also be stratified according to boundary activation levels. Functions within a system need not become active simultaneously on configuration points. Instead, subsets of constraints may activate on different geometric strata of the carrier. Such analytic stratification parallels geometric decompositions studied earlier, where hyperplanes or surfaces generated layered partitions of configurations.

In functional terms, stratification corresponds to hierarchical activation of inequality systems defining carrier boundaries.

Another important classification concerns smoothness properties of defining functions. Representation systems constructed from differentiable convex functions yield carriers admitting tangent structure and differential boundary analysis. Nondifferentiable systems, by contrast, produce carriers with angular or polyhedral features. Thus analytic regularity of representation systems governs geometric regularity of configuration carriers.

Functional carrier classification also reflects dimensional embedding of configurations. High-dimensional representation systems may encode low-dimensional configurations through redundant or degenerate constraints, whereas minimal-dimensional systems achieve greater analytic economy.

Dimensional considerations therefore influence structural efficiency of representation systems. These analytic classifications show that functional representations form not a single homogeneous framework but a hierarchy of carrier models ranging from polyhedral to smooth convex and hybrid analytic geometries. Each class preserves extremal configuration structure while providing distinct analytic tools for describing carrier domains.

An important methodological consequence of this universality lies in the possibility of transferring structural results between geometric and analytic frameworks. Extremal properties established in polyhedral geometry admit analytic reformulation through affine inequality systems, while curvature-induced extremality may be represented through strictly convex carrier functions. Such transfers permit analytic investigation of geometric configuration properties.

The analytic apparatus developed in this chapter thus completes the representation-theoretic description of point configurations. Discrete sets embedded in $\mathbb{R}^n$ have been encoded through systems of functional inequalities whose carriers reproduce convex hull structure, supporting geometry, and surface embedding of the configuration. Polyhedral, smooth convex, and hybrid analytic carrier classes emerge naturally from this encoding, each preserving extremal configuration structure while providing distinct analytic tools for describing carrier domains. This hierarchy of carrier models constitutes a unified analytic language capable of describing the full range of geometric structures associated with finite point configurations.

The representation framework established here completes the geometric and analytic foundations of the monograph. Vertex-located configurations, carrier polytopes, decomposition structures, and functional representation systems together constitute a unified language in which discrete combinatorial geometry is encoded within continuous convex analytic frameworks. These foundations are directly exploited in Chapter 2, where functions defined on vertex-located configurations are extended to convex carrier domains, and in Chapter 3, where concrete combinatorial configuration classes are introduced and their polyhedral geometry is systematically developed.

Chapter 2

Convex Extensions of Functions on Finite Point Configurations

Functions defined on finite point configurations arise naturally in combinatorial optimization, where objective functions are evaluated at discrete feasible points rather than on continuous domains. The passage from discrete to continuous analytic settings requires extension constructions that preserve the extremal structure of the original problem. This chapter develops a systematic theory of convex extensions for functions defined on finite vertex-located point configurations, establishing the analytic core of the monograph.

The extension problem is formulated precisely in §2.1, where three principal classes of extensions are introduced: convex, strongly convex, and differentiable, each defined by progressively stronger regularity conditions on the carrier domain. The vertex-located property, meaning the coincidence of the configuration with the vertex set of its convex hull, plays a decisive role throughout: it guarantees that convexity imposes no internal compatibility constraints on prescribed functional data, so that extensions exist for arbitrary functions.

Existence theorems for all three classes are proved in §2.2. The minimal convex extension is constructed via extremal interpolation over vertex decompositions of the carrier polyhedron. Differentiable extensions are obtained through a deformation and Minkowski functional construction, while strongly convex extensions are built by adding a spherical correction term with an arbitrary curvature parameter. The structural and geometric properties of convex extensions are analyzed in §2.3, including epigraph polyhedrality of minimal extensions and two-sided interpolation bounds that confine all admissible extensions within the carrier domain. Lower bounds for the minimum of a function over a vertex-located configuration are derived in §2.4 from analytic properties of its convex extension, and sufficient optimality conditions are formulated in terms of the gradient structure of the extension. The chapter concludes with convex extensions on hyperspheres in §2.5, where strongly convex differentiable extensions are constructed via a penalty approach exploiting the spherical carrier geometry.

Together, the existence theorems, structural properties, and extremal estimates established here are applied directly to the concrete configuration classes introduced in Chapter 3 and to the optimization methods of Chapter 5.

§2.1. Functions on Finite Point Configurations and Convex Extension Framework

The study of functions defined on finite point configurations and their continuation to convex carrier domains occupies a central position in convex analysis and its applications to discrete geometry and optimization. When functional data are prescribed at a finite collection of points in $\mathbb{R}^n$, it is natural to ask under what conditions such data admit an extension to a continuous convex function defined on a domain containing the configuration, and what analytic and geometric properties such extensions may possess. The answer depends essentially on the geometric structure of the configuration itself, in particular on whether the configuration points occupy extremal positions within their convex hull.

Throughout this chapter we consider finite point configurations

$$E = \{x^1 \dots, x^m\} \subset \mathbb{R}^n$$

that are vertex-located, meaning that the configuration coincides with the vertex set of its convex hull:

$$E = \text{vert conv } E.$$

No point of E may therefore be expressed as a convex combination of the remaining points. As will be seen, this extremality property plays a decisive role in extension theory: it ensures that convexity imposes no internal compatibility constraints on prescribed functional data, and that extensions exist for arbitrary functions defined on the configuration.

Suppose that a function $\varphi : E \rightarrow \mathbb{R}$ is prescribed on the configuration. The values $\varphi(x^i)$ may encode geometric weights, combinatorial costs, energetic characteristics, or optimization parameters associated with configuration points. The central problem of this chapter is the following: construct a function Φ defined on a convex domain containing $\text{conv } E$ that agrees with φ at every configuration point and satisfies prescribed analytic regularity conditions throughout the domain.

The nature of these regularity conditions determines the class of admissible extensions. Three principal classes will be studied: general convex extensions, strongly convex extensions, and differentiable convex extensions. Each class is built upon a corresponding notion of functional regularity, and we now recall these notions in the form required for the extension constructions that follow.

A function $\Phi : D \rightarrow \mathbb{R}$ defined on a convex set $D \subset \mathbb{R}^n$ is called convex if for all $x, y \in D$ and all $\lambda \in (0, 1)$,

$$\Phi(\lambda x + (1 - \lambda)y) \leq \lambda \Phi(x) + (1 - \lambda) \Phi(y).$$

Geometrically, this inequality requires that the chord connecting any two points on the graph of Φ lies above or on the graph. Equivalently, the epigraph of Φ , defined as

$$\text{epi } \Phi = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : t \geq \Phi(x)\},$$

is a convex set in $\mathbb{R}^{n+1}$.

Convex functions need not be differentiable; at points of nondifferentiability, the gradient is replaced by the subdifferential $\partial\Phi(x)$, whose elements are called subgradients and represent slopes of supporting hyperplanes to the epigraph at the point $(x, \Phi(x))$.

With this notion in place, the first class of extensions is defined as follows. Let $E \subset \mathbb{R}^n$ be a finite vertex-located configuration, let $\varphi : E \rightarrow \mathbb{R}$ be a prescribed function, and let $D \subset \mathbb{R}^n$ be a convex set satisfying $\text{conv } E \subset D$.

Definition 2.1. A function $\Phi : D \rightarrow \mathbb{R}$ is called a convex extension of φ from E to D if Φ is convex on D and $\Phi(x) = \varphi(x)$ for every $x \in E$. The set D is called a convex carrier domain for the extension.

The minimal convex carrier domain is $\text{conv } E$ itself, generated directly by the extremal structure of E . On this minimal carrier, the boundary values of any convex extension are fixed at the vertices of the polyhedral hull, and extension behavior in the interior is governed by convex interpolation between vertex data. While $\text{conv } E$ serves as the natural domain for continuation constructions, extensions may also be defined on larger convex superdomains $D \supseteq \text{conv } E$, a possibility to which we return below.

General convex extensions need not possess curvature or smoothness. In many analytical and optimization contexts, stronger regularity is required. The appropriate strengthening is provided by the notion of strong convexity.

A function $\Phi : D \rightarrow \mathbb{R}$ is called strongly convex with parameter $\mu > 0$ if for all $x, y \in D$ and all $\lambda \in (0, 1)$,

$$\Phi(\lambda x + (1 - \lambda)y) \leq \lambda\Phi(x) + (1 - \lambda)\Phi(y) - \mu\lambda(1 - \lambda)\|x - y\|^2.$$

The additional quadratic term $\mu\lambda(1 - \lambda)\|x - y\|^2$ measures the uniform separation between the function and its affine interpolants, and may be interpreted as a curvature lower bound: the graph of Φ curves away from every supporting hyperplane at a rate controlled by μ . Strong convexity implies strict convexity and ensures that sublevel sets are strictly convex and that minimizers, when they exist, are unique.

This curvature structure motivates the second class of extensions.

Definition 2.2. The extension Φ is called a μ -strongly convex extension of φ on D if Φ is strongly convex with parameter $\mu > 0$ on D and $\Phi(x) = \varphi(x)$ for every $x \in E$.

Strong convexity guarantees curvature throughout the carrier domain but does not in itself ensure smoothness. A strongly convex function may still be

nondifferentiable along certain directions, retaining polyhedral features inherited from the underlying convex structure. When the application requires smooth continuation – for instance, in gradient-based optimization or variational analysis – an additional differentiability condition must be imposed.

Recall that $C^k(D)$ denotes the class of functions possessing continuous derivatives up to order k on D ; in particular, $C^1(D)$ requires continuity of all first-order partial derivatives, and $C^2(D)$ additionally requires continuity of all second-order partial derivatives. A convex function $\Phi : D \rightarrow \mathbb{R}$ belonging to $C^1(D)$ possesses a continuous gradient

$$\Phi' = \left(\frac{\partial \Phi(x)}{\partial x_1}, \dots, \frac{\partial \Phi(x)}{\partial x_n} \right)$$

on D . In this case, convexity is equivalent to the first-order gradient inequality

$$\Phi(y) \geq \Phi(x) + \langle \Phi'(x), y - x \rangle \quad \forall x, y \in D,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$.

This inequality states that the function lies above every tangent hyperplane.

For functions in $C^2(D)$, convexity is additionally characterized by positive semidefiniteness of the Hessian $\Phi''(x)$ at every $x \in D$. Differentiability eliminates the subdifferential multiplicity present in general convex functions: at every interior point $x \in D$, the subdifferential reduces to the singleton $\partial\Phi(x) = \{\Phi'(x)\}$.

These properties naturally lead to the third class of extensions.

Definition 2.3. The extension Φ is called a differentiable convex extension of φ on D if $\Phi \in C^k(D)$ for some $k \geq 1$ and $\Phi(x) = \varphi(x)$ for every $x \in E$.

The three definitions introduced above share a common interpolation requirement agreement with φ on E while imposing progressively stronger analytic conditions on the carrier domain.

Together they generate a hierarchy of extension classes reflecting increasing regularity: from polyhedral convex surfaces through curvature-controlled strongly convex extensions to smooth differentiable realizations.

Minimal carriers enforce purely geometric extension structure determined by configuration extremality, while supercarriers introduce analytic degrees of freedom beyond the convex hull, enabling compatibility with global geometric constraints of the ambient domain.

A fundamental constraint governing all convex extensions is the interpolation inequality.

Every interior point $x \in \text{conv } E$ admits a convex decomposition

$$x = \sum_{i=1}^m \lambda_i x^i, \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1.$$

Convexity of Φ then imposes the upper bound

$$\Phi(x) \leq \sum_{i=1}^m \lambda_i \varphi(x^i),$$

which constrains admissible extension values at every interior carrier point.

Conversely, every convex extension is bounded below by the family of affine functions whose values at configuration points do not exceed the prescribed data.

The interaction of these upper interpolation bounds and lower affine supports generates a continuation admissibility region within which all convex extensions compatible with φ must lie.

Since E is vertex-located, this region is finitely generated by the vertex system and imposes no internal consistency requirements on the prescribed values – a structural freedom that makes vertex-located configurations the natural and essentially optimal setting for convex extension theory.

§2.2. Existence of Convex Extensions on Vertex-Located Configurations

We now proceed from the general extension framework introduced in §2.1 to the constructive theory of convex extensions on vertex-located point configurations.

Let $E = \{x^1, x^2, \dots, x^m\} \subset \mathbb{R}^n$ be a finite vertex-located configuration, and let $\varphi : E \rightarrow \mathbb{R}$ be a prescribed function. Let $\text{conv } E$ denote the convex hull of the configuration, which serves as the minimal convex carrier domain for admissible extensions.

The set of all convex decompositions of a point $x \in \text{conv } E$ with respect to the vertex system of $\text{conv } E$ is denoted

$$\Lambda(x) = \left\{ (\lambda^1, \dots, \lambda_m) : x = \sum_{i=1}^m \lambda_i x^i, \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1 \right\}.$$

The set $\Lambda(x)$ is nonempty for every $x \in \text{conv } E$, convex and compact as a polytope in $\mathbb{R}^m$, and contains a decomposition with affinely independent active vertices whenever x lies in the interior of a face of $\text{conv } E$. Three classes of extensions will be constructed: convex, differentiable convex, and strongly convex.

Theorem 2.1. (*Existence of Convex Extensions on Vertex-Located Configurations*).

Let $E \subset \mathbb{R}^n$ be a finite vertex-located configuration and let $\varphi : E \rightarrow \mathbb{R}$ be a prescribed function. Then there exists a convex function $\Phi : \text{conv } E \rightarrow \mathbb{R}$ satisfying $\Phi(x) = \varphi(x)$ for every $x \in E$.

Proof. Define the extension function Φ on $\text{conv } E$ by

$$\Phi(x) = \min \left\{ \sum_{i=1}^m \lambda_i \varphi(x^i) : (\lambda_1, \dots, \lambda_m) \in \Lambda(x) \right\}. \quad (2.1)$$

That is, $\Phi(x)$ is the minimal convex combination of vertex values compatible with all convex decompositions of x relative to the vertex system of $\text{conv } E$.

We first verify that Φ is well defined. Since $x \in \text{conv } E$, the set $\Lambda(x)$ is nonempty. The functional expression

$$\sum_{i=1}^m \lambda_i \varphi(x^i)$$

is linear in λ , and $\Lambda(x)$ is compact, so the minimum is attained.

Next we show that Φ extends φ . If $x = x^k$ is a vertex of $\text{conv } E$, then the unique element of $\Lambda(x^k)$ is the coefficient system $\lambda_k = 1$ and $\lambda_i = 0$ for $i \neq k$. Therefore $\Phi(x^k) = \varphi(x^k)$, and hence $\Phi(x) = \varphi(x)$ for every $x \in E$.

Finally we establish convexity. Let $x, y \in \text{conv } E$ and $\alpha \in (0, 1)$. Let $\{\lambda_i^*\} \in \Lambda(x)$ and $\{\mu_i^*\} \in \Lambda(y)$ be optimal coefficient systems achieving

$$\Phi(x) = \sum_{i=1}^m \lambda_i^* \varphi(x^i)$$

$$\Phi(y) = \sum_{i=1}^m \mu_i^* \varphi(x^i).$$

Set $v_i = \alpha \lambda_i^* + (1 - \alpha) \mu_i^*$. Then

$$\sum_{i=1}^m v_i = 1, v_i \geq 0,$$

$$\alpha x + (1 - \alpha) y = \sum_{i=1}^m v_i x^i,$$

so $\{v_i\} \in \Lambda(\alpha x + (1 - \alpha)y)$.

Substituting into formula (2.1) yields

$$\begin{aligned} \Phi(\alpha x + (1 - \alpha)y) &\leq \sum_{i=1}^m v_i \varphi(x^i) = \\ &= \alpha \sum_{i=1}^m \lambda_i^* \varphi(x^i) + (1 - \alpha) \sum_{i=1}^m \mu_i^* \varphi(x^i) = \\ &= \alpha \Phi(x) + (1 - \alpha) \Phi(y). \end{aligned}$$

Thus Φ is convex on $\text{conv } E$. ■

Remark 2.1. The vertex-located assumption plays a decisive role in ensuring unrestricted feasibility of convex extensions. If a point $x^k \in E$ is not a vertex of $\text{conv } E$, then x^k admits a convex representation in terms of other configuration points, and convexity of any extension Φ satisfying $\Phi(x) = \varphi(x)$ for every $x \in E$ imposes the compatibility condition

$$\varphi(x^k) \leq \sum_{\substack{i=1 \\ i \neq k}}^m \lambda_i \varphi(x^i).$$

In contrast, vertex-located configurations eliminate all such constraints: since every point of E is an extremal point of $\text{conv } E$, no nontrivial convex representation of any configuration point in terms of the others exists, and convex extensions satisfying the interpolation condition exist for arbitrary functions $\varphi: E \rightarrow \mathbb{R}$. Vertex-located configurations therefore constitute the maximal class of finite point sets admitting unrestricted convex extension.

The extension Φ constructed in Theorem 2.1 is the minimal convex extension of φ : its epigraph coincides with the convex hull of the lifted point set

$$\{(x^i, \varphi(x^i))\} \subset \mathbb{R}^{n+1},$$

so Φ is a polyhedral convex function whose graph consists of affine facets interpolating between vertex values.

While the minimal convex extension of Theorem 2.1 preserves the polyhedral structure of the carrier, many analytical applications require extension functions possessing smoothness. This motivates the construction of differentiable convex extensions satisfying the requirements of Definition 2.3.

Theorem 2.2 (Existence of Globally Smooth Convex Extensions). Let $E = \{x^1, \dots, x^N\} \subset \mathbb{R}^n$ be a finite vertex-located configuration, and let $\varphi: E \rightarrow \mathbb{R}$ be arbitrary. Then there exists a convex function $\Phi: \mathbb{R}^n \rightarrow \mathbb{R}$ of class $C^\infty(\mathbb{R}^n)$ satisfying

$$\Phi(x^i) = \varphi(x^i), \quad i = 1, \dots, N.$$

Proof. Since every x^i is a vertex of $\text{conv } E$, it is strictly exposed: there exists $v_i \in \mathbb{R}^n$ such that $\langle v_i, x^i - x^j \rangle > 0$ for all $j \neq i$. Choose a common number $t > 0$ sufficiently large that the affine functions

$$L_i(x) = \varphi(x^i) + t \langle v_i, x - x^i \rangle, \quad i = 1, \dots, N.$$

satisfy $L_i(x^j) < \varphi(x^j)$ whenever $i \neq j$. Such a choice is possible because there are finitely many pairs (i, j) and the scalar products have strictly negative signs when evaluated as $\langle v_i, x^j - x^i \rangle$.

Define the convex function

$$F(x) = \max_{1 \leq i \leq N} L_i(x), \quad x \in \mathbb{R}^n.$$

At each x^i , the unique maximizing function is L_i , and hence $F(x^i) = \varphi(x^i)$. The strict inequalities and continuity imply that $F = L_i$ in some open neighbourhood U_i of x^i .

Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a nonnegative, radially symmetric function with $\int_{\mathbb{R}^n} \eta(u) du = 1$ and zero first moment. For $\delta > 0$ set $\eta_\delta(u) = \delta^{-n} \eta(u/\delta)$ and define

$$\Phi(x) = (F * \eta_\delta)(x) = \int_{\mathbb{R}^n} F(x-u) \eta_\delta(u) du. \quad (2.2)$$

The convolution is finite everywhere and belongs to $C^\infty(\mathbb{R}^n)$. It is convex because the convexity inequality for F can be integrated against the nonnegative kernel η_δ .

Choose δ sufficiently small that $x^i - \text{supp } \eta_\delta \subset U_i$ for every i . Since $F = L_i$ on U_i , the unit mass and zero first moment of η_δ give

$$\Phi(x^i) = \int_{\mathbb{R}^n} L_i(x^i - u) \eta_\delta(u) du = L_i(x^i) = \varphi(x^i).$$

Thus Φ is a globally defined, infinitely differentiable convex extension of φ . ■

Remark (on smoothness). Theorem 2.2 gives a globally smooth convex extension. Its restriction to $\Pi = \text{conv } E$ is continuously differentiable in the sense of Definition 2.3. Smoothness does not imply strong convexity.

Geometrically, the proof constructs a polyhedral convex interpolant as the maximum of affine functions, each strictly active near its associated vertex. Smoothing removes the nonsmooth ridges while preserving the vertex values.

While smooth extensions provide derivatives of every order, they need not satisfy a uniform positive curvature condition. Many optimization applications require strong convexity with a positive curvature parameter ρ . This motivates the construction of ρ -strongly convex extensions in the sense of Definition 2.2.

Theorem 2.3. (*Existence of Strongly Convex Extensions*).

Let $E \subset \mathbb{R}^n$ be a finite vertex-located configuration, let $\varphi : E \rightarrow \mathbb{R}$ be a prescribed function, and let $\Pi = \text{conv } E$. Then for any $\rho > 0$ there exists a ρ -strongly convex function $\Psi_\rho : \Pi \rightarrow \mathbb{R}$ satisfying $\Psi_\rho(x) = \varphi(x)$ for every $x \in E$.

Proof. Let Φ denote the minimal convex extension of φ constructed in Theorem 2.1. The strategy is to construct an auxiliary function $h : \Pi \rightarrow \mathbb{R}$ that is strongly convex with parameter 1 and vanishes on E . Then

$$\Psi_\rho(x) = \Phi(x) + \rho h(x)$$

will be ρ -strongly convex and coincide with $\varphi(x)$ on E . To construct $h(x)$ via spherical projection, choose a radius $r > 0$ such that the open ball $B_r(0)$ of radius r centered at the origin contains Π , so that $\Pi \subset B_r(0)$.

Denote by $S_r(0) = \partial B_r(0)$ the corresponding sphere.

For each vertex $x^i \in E$ define its orthogonal projection onto the sphere:

$$b^i = \text{Pr}_{S_r(0)} x^i = r x^i / \|x^i\|, \quad i \in J_m,$$

where $J_m = \{1, \dots, m\}$ indexes the vertices of E .

Now define the auxiliary function

$$h(x) = \max_{i \in J_m} \left\{ \|x - x^i + b^i\|^2 - \|b^i\|^2 \right\}.$$

Each term $\|x - x^i + b^i\|^2 - \|b^i\|^2$ is a quadratic function of x with Hessian $2I$, hence is strongly convex with parameter 1. Since $h(x)$ is the pointwise maximum of finitely many such functions, it is strongly convex with parameter 1 on Π . To verify that $h(x)$ vanishes on E , we compute the i -th term of h at $x = x^k$:

$$\|x^k - x^i + b^i\|^2 - \|b^i\|^2 = \|x^k - x^i\|^2 + 2\langle x^k - x^i, b^i \rangle.$$

For $i = k$ the expression vanishes:

$$\|b^k\|^2 - \|b^k\|^2 = 0.$$

For $i \neq k$, the expression equals

$$\|x^k - x^i\|^2 + 2\langle x^k - x^i, b^i \rangle.$$

Hence $h(x^k) = 0$ for every $x^k \in E$, provided r is chosen sufficiently large.

We now set

$$\Psi_\rho(x) = \Phi(x) + \rho h(x).$$

Since Φ is convex and ρh is ρ -strongly convex, their sum Ψ_ρ is ρ -strongly convex on Π .

Since $h(x^k) = 0$ for all $x^k \in E$ and $\Phi(x^k) = \varphi(x^k)$, we have $\Psi_\rho(x^k) = \varphi(x^k)$ for every vertex, so the interpolation condition holds on E .

The construction extends naturally to any convex superdomain $D \supseteq \Pi$ by extending Φ and h beyond Π while preserving convexity. The spherical projection construction establishes the connection with the spherical carrier geometry studied in §1.3. This completes the proof. ■

The three theorems proved above demonstrate that convex extensions on vertex-located configurations may be realized across a full regularity spectrum. All three constructions preserve the interpolation condition $\Phi(x) = \varphi(x)$ for every $x \in E$ on the minimal carrier $\Pi = \text{conv } E$, and each admits continuation to any convex superdomain $D \supseteq \Pi$. The structural and extremal properties of these extension classes are investigated in the following sections.

Remark 2.2. A strongly convex extension may also be obtained through affine correction of the minimal convex extension. Let Φ be the minimal convex extension of φ from Theorem 2.1. Define

$$\Psi_\rho(x) = \Phi(x) + \rho \|x\|^2 - a(x),$$

where $a(x) = \langle c, x \rangle + d$ is an affine function determined by the interpolation conditions $\langle c, x^i \rangle + d = \rho \|x^i\|^2$ for all $x^i \in E$.

Such a system is consistent because $\rho\|x^i\|^2$ decomposes into an affine part and a quadratic correction that is balanced by the affine freedom on $\text{aff } E$. Since Φ is convex and $\rho\|x\|^2$ is ρ -strongly convex, subtracting the affine function $a(x)$ preserves strong convexity, and the interpolation condition

$$\Psi_\rho(x^i) = \varphi(x^i)$$

holds on E by construction. This alternative construction complements the spherical projection approach of Theorem 2.3 and is particularly convenient when an explicit formula for the correction term is preferred.

The constructions established in the preceding blocks demonstrate that convex extensions on vertex-located configurations exist in multiple analytic regimes, ranging from minimal polyhedral extensions to strongly convex and differentiable extensions. These extension classes form a structural spectrum reflecting increasing analytic regularity while preserving the interpolation condition imposed by configuration vertices.

An essential aspect of extension theory concerns the dependence of admissible extensions on the choice of convex carrier domain. Thus far, extension constructions have been developed primarily on the minimal carrier $\text{conv } E$ generated by the configuration itself.

However, as noted in §2.1, convex extensions may be constructed on larger convex superdomains $D \supseteq \text{conv } E$, provided that convexity and interpolation conditions are preserved.

When extensions are realized on minimal carriers, continuation structure is governed entirely by the polyhedral geometry of $\text{conv } E$. Minimal convex extensions inherit facet stratifications of the hull, strongly convex extensions introduce curvature within polyhedral boundaries, and smooth extensions regularize polyhedral discontinuities while remaining confined to the hull.

In contrast, extension constructions on convex supercarriers admit additional analytic flexibility. Outside $\text{conv } E$, extension values are no longer constrained by vertex interpolation and may be shaped to satisfy global convexity or domain-specific geometric requirements.

For example, if $\text{conv } E$ is embedded within a convex cone or orthant domain, one may construct convex extensions compatible with monotonicity constraints or domain feasibility conditions arising in optimization contexts. Such supercarrier extensions preserve convexity globally while maintaining interpolation fidelity on the configuration.

The distinction between minimal-carrier and supercarrier extensions reveals an important structural dichotomy:

- minimal carriers enforce purely geometric extension structure determined by configuration extremality;
- supercarriers introduce analytic degrees of freedom beyond the convex hull.

This dichotomy plays a significant role in extension-based optimization frameworks, where feasible domains often extend beyond minimal configuration carriers.

The extension classes constructed above may therefore be compared along two principal dimensions:

- regularity of the extension (polyhedral, strongly convex, smooth);
- extent of the carrier domain (minimal hull vs. supercarrier).

Together, these dimensions generate an extension taxonomy describing admissible convex extensions on vertex-located configurations.

From the standpoint of extremal analysis, the existence of multiple extension classes raises fundamental questions concerning the behavior of convex extensions within carrier domains.

In particular, one may ask:

- how extension values are bounded relative to vertex data;
- how curvature affects extremal growth rates;
- how supporting hyperplanes constrain admissible extensions;
- how gradient structure reflects configuration geometry.

These questions lead naturally to the study of extremal properties of convex extensions.

The existence theory developed in this section establishes that convex extensions on vertex-located configurations may be constructed in minimal, strongly convex, and differentiable analytic regimes while preserving prescribed functional values at all configuration points.

Beyond mere realizability, these constructions reveal a deeper geometric phenomenon. Since vertex-located configurations eliminate internal convex compatibility constraints, admissible extension behavior becomes entirely governed by extremal carrier geometry.

In particular, continuation values, curvature growth, and gradient structure of convex extensions are constrained by vertex interpolation bounds, supporting hyperplanes of the convex hull, and normal cone systems induced by carrier extremality. Thus, vertex location does not merely guarantee existence of convex extensions; it rigidly determines their admissible functional and variational behavior within the carrier domain.

The extremal analysis of convex extensions therefore investigates functional bounds, support structures, and optimality conditions determined jointly by vertex interpolation constraints and convex carrier geometry. Minimal extensions produce sharp interpolation bounds, strongly convex extensions generate quadratic growth estimates, and smooth extensions admit differential extremal characterizations.

Thus, the existence theory developed in this section provides the constructive foundation upon which extremal functional analysis on vertex-located configurations is built. Having established the realizability of convex extensions across

multiple regularity regimes, we now proceed to investigate their extremal behavior, including lower bounds, gradient estimates, and sufficient optimality conditions defined relative to convex carrier domains.

§2.3. Structural Properties and Geometric Interpretation of Convex Extensions

Having established the existence of convex, strongly convex, and differentiable extensions on vertex-located configurations, we now proceed to investigate their structural properties. The objective of this section is to analyze how the extremal geometry of the configuration determines the analytic and geometric structure of admissible convex extensions.

Let $E = \{x^1, x^2, \dots, x^m\} \subset \mathbb{R}^n$ be a vertex-located configuration, let $\varphi : E \rightarrow \mathbb{R}$ be a prescribed function, and let $\Phi : \text{conv } E \rightarrow \mathbb{R}$ be a convex extension of φ in the sense of Definition 2.1.

A fundamental geometric object associated with Φ is its epigraph,

$$\text{epi } \Phi = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : t \geq \Phi(x)\}.$$

The epigraph provides a lifted convex carrier encoding continuation structure in one higher dimension. Properties of Φ may therefore be studied through the convex geometry of $\text{epi } \Phi$.

For minimal convex extensions constructed via extremal interpolation, the epigraph admits a particularly simple structure.

Theorem 2.4. (*Polyhedral Structure of Minimal Convex Extensions*).

Let Φ be the minimal convex extension of φ constructed as in Theorem 2.1. Then $\text{epi } \Phi$ is a polyhedral convex set in $\mathbb{R}^{n+1}$ whose extreme points are contained in the lifted configuration $\{(x^i, \varphi(x^i))\}$.

Proof. By construction, $\Phi(x)$ is defined as the minimal convex interpolation compatible with vertex values. Consequently, $\text{epi } \Phi$ coincides with the convex hull of the lifted point set together with its upward vertical extension.

Convex hulls of finite point sets in $\mathbb{R}^{n+1}$ are polyhedral. Hence $\text{epi } \Phi$ is polyhedral. ■

Convex extension operators constructed above admit a geometric interpretation as value propagation mechanisms defined over carrier polyhedra. Interpolation values prescribed at vertex configurations extend into the carrier interior through convex envelope formation, generating admissible continuation domains constrained by supporting hyperplanes. Thus, convex extension may be viewed as a structural lifting of discrete functional data into continuous convex geometry.

This result establishes that minimal convex extensions preserve the combinatorial structure of the configuration under epigraph lifting. Facets of $\text{epi } \Phi$ correspond to affine interpolants determined by subsets of configuration points.

Thus, the polyhedral stratification of $\text{conv } E$ is mirrored by facet decomposition of the extension epigraph.

The geometry of supporting hyperplanes to $\text{epi } \Phi$ plays a central role in extension analysis. Each supporting hyperplane corresponds to an affine function

$$\ell(x) = \langle a, x \rangle + b$$

satisfying $\ell(x^i) \leq \varphi(x^i)$ for all i , and coinciding with Φ on subsets of $\text{conv } E$.

Such affine supports determine local extension behavior and govern interpolation regions within the carrier domain.

Interior points of $\text{conv } E$ are associated with convex combinations of vertex supports. Consequently, extension values at interior locations are determined by convex envelopes of affine interpolants generated by subsets of configuration points.

This mechanism reveals that minimal convex extensions propagate vertex data into the interior via supporting hyperplane interpolation.

The polyhedral structure of minimal extensions also determines differentiability properties. At points where multiple affine supports coincide, Φ fails to be differentiable. These nondifferentiable loci correspond to intersections of facets in the epigraph and form polyhedral stratification subsets of $\text{conv } E$.

Thus, nondifferentiability of minimal extensions is not accidental but reflects intrinsic combinatorial structure of the configuration hull.

Strongly convex extensions modify this polyhedral geometry by introducing curvature into the epigraph structure. Facets are replaced by strictly convex surfaces, eliminating degeneracies in supporting hyperplane systems.

For any strongly convex extension Φ , $\text{epi } \Phi$ becomes a strictly convex set in $\mathbb{R}^{n+1}$, admitting unique supporting hyperplanes at all interior points. Strong convexity constructions introduced above may be interpreted as regularized convex extensions.

Quadratic regularization stabilizes continuation behavior, eliminating flat interpolation regions and ensuring uniqueness of admissible continuation gradients. This regularization perspective reveals that curvature parameters serve not only analytic but also stability-inducing roles in extension theory.

This structural regularization has important consequences for gradient behavior and extremal functional estimates considered in subsequent sections.

Differentiable convex extensions further refine epigraph geometry by replacing piecewise-linear or curved-but-nondifferentiable surfaces with smooth convex envelopes. Their supporting hyperplanes vary continuously, and gradient fields define continuous mappings from $\text{conv } E$ into dual space.

Thus, smooth convex extensions provide analytic carriers suitable for differential extremal analysis.

The geometric interpretation developed above demonstrates that continuation structure is encoded not only in functional values but in the lifted convex geometry of epigraph carriers.

Vertex extremality, hull polyhedrality, and extension regularity jointly determine the stratification, curvature, and differentiability structure of extension epigraphs.

The polyhedral geometry of minimal convex extensions determines not only the structure of their epigraphs but also the interpolation mechanisms governing extension values throughout the carrier domain.

Let Φ be the minimal convex extension of φ on $\text{conv } E$ constructed as in Theorem 2.1. For any point $x \in \text{conv } E$, consider an arbitrary convex decomposition

$$x = \sum_{i=1}^m \lambda_i x^i, \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1.$$

Convexity of Φ implies the interpolation inequality

$$\Phi(x) \leq \sum_{i=1}^m \lambda_i \varphi(x^i).$$

Thus, extension values at interior carrier points are bounded above by affine interpolants generated by vertex data.

This inequality holds for every convex extension of φ , not only for minimal ones.

Theorem 2.5. (*Convex Interpolation Bound*).

Let Φ be any convex extension of φ on $\text{conv } E$. Then for all $x \in \text{conv } E$ and for every convex decomposition of x ,

$$\Phi(x) \leq \sum_{i=1}^m \lambda_i \varphi(x^i).$$

Proof. The result follows directly from convexity of Φ and the definition of convex combinations.

The interpolation inequality defines an upper admissibility envelope governing extension values inside the convex hull. Minimal convex extensions attain the smallest values compatible with this constraint structure. ■

In particular, minimal extensions coincide with the lower convex envelope of all affine interpolants determined by vertex data.

The extension structure may therefore be described through an extremal interpolation principle: extension values at interior points arise as extremal convex combinations of boundary data consistent with carrier geometry.

This principle reflects the propagation of discrete functional values into continuous domains via convex hull interpolation.

A dual description may be formulated in terms of supporting affine minorants. Let

$$\ell(x) = \langle a, x \rangle + b$$

be an affine function satisfying $\ell(x^i) \leq \varphi(x^i)$ for all i . Then convexity implies $\ell(x) \leq \Phi(x)$ for all $x \in \text{conv } E$.

Thus, every convex extension is bounded below by the family of affine supports compatible with vertex data.

Combining upper interpolation bounds and lower affine supports yields a two-sided admissibility structure:

$$\ell(x) \leq \Phi(x) \leq \sum_{i=1}^m \lambda_i \varphi(x^i).$$

Minimal convex extensions arise precisely when these bounds coincide as tightly as convexity permits.

This dual bounding mechanism provides a complete geometric characterization of admissible convex extensions. Extension values are confined between polyhedral interpolation envelopes determined solely by configuration extremality.

No admissible convex extension may violate these interpolation bounds without contradicting convexity or vertex interpolation conditions.

The interpolation structure described above reveals that the convex hull $\text{conv } E$ acts simultaneously as:

- a geometric carrier domain;
- an interpolation skeleton;
- an extremal support system governing extension admissibility.

The extremality of vertex-located configurations ensures that interpolation bounds are finitely generated and combinatorially structured.

Strongly convex extensions modify this interpolation geometry by introducing curvature between interpolation bounds. While satisfying the same vertex constraints, strongly convex extensions deviate quadratically from affine interpolants, producing strictly interior extension surfaces.

Smooth convex extensions further refine this structure by eliminating polyhedral nondifferentiabilities while remaining confined within admissibility envelopes.

Thus, all convex extension classes share a common interpolation backbone determined by vertex data, while differing in curvature and regularity of admissible realizations.

This structural unity underlies extremal estimates derived in subsequent sections.

The interpolation structure of convex extensions determines not only admissible functional values but also the differential and variational geometry of extension surfaces. This structure is most naturally described through subgradients and normal cones associated with the carrier polyhedron.

Let Φ be a convex extension of φ defined on $\text{conv } E$. At each point $x \in \text{conv } E$ one defines the subdifferential set

$$\partial\Phi(x) = \{g \in \mathbb{R}^n : \Phi(y) \geq \Phi(x) + \langle g, y - x \rangle\}$$

for all $y \in \text{conv } E$.

Elements of $\partial\Phi(x)$ represent supporting hyperplane slopes to the epigraph of Φ at x .

The geometry of $\partial\Phi(x)$ reflects the local interpolation structure induced by vertex data.

If x lies in the interior of $\text{conv } E$ and Φ is differentiable at x , then

$$\partial\Phi(x) = \{\Phi'(x)\}$$

reduces to a singleton gradient vector.

However, if Φ is nondifferentiable at x , the subdifferential becomes a convex set representing multiple admissible supporting slopes.

Such nondifferentiability arises naturally in minimal convex extensions along facet intersections of the extension epigraph.

At configuration vertices $x^i \in E$, the subdifferential structure is determined jointly by extension geometry and polyhedral carrier extremality.

Theorem 2.6. (*Vertex Subdifferential Structure*).

Let Φ be a convex extension of φ on $\text{conv } E$. Then for each vertex $x^i \in E$, $\partial\Phi(x^i)$ contains the normal cone of $\text{conv } E$ at x^i .

Proof. Since x^i is an extremal point of $\text{conv } E$, every supporting hyperplane to $\text{conv } E$ at x^i defines a direction of outward normality. Convexity of Φ and the restriction condition $\Phi(x^i) = \varphi(x^i)$ imply that supporting hyperplanes of the carrier induce admissible supporting hyperplanes of the epigraph at the lifted point $(x^i, \varphi(x^i))$.

Hence the normal cone of $\text{conv } E$ embeds into $\partial\Phi(x^i)$. ■

This result demonstrates that carrier extremality directly influences the variational structure of convex extensions. Subgradients at configuration vertices inherit geometric constraints imposed by polyhedral hull structure.

Thus, vertex-located configurations transmit extremal geometry into differential properties of admissible extensions.

For minimal convex extensions, subdifferential sets admit a combinatorial interpretation. Each element of $\partial\Phi(x)$ corresponds to an affine interpolant determined by subsets of configuration points generating local supporting facets of the epigraph.

Hence subgradient multiplicity reflects the number of active interpolation supports at the given carrier point.

Strongly convex extensions exhibit fundamentally different subdifferential behavior.

Theorem 2.7. (*Subgradient Uniqueness for Strongly Convex Extensions*).

Let Φ be a ρ -strongly convex extension of φ . Then Φ is differentiable on the interior of $\text{conv } E$, and for every interior point x ,

$$\partial\Phi(x) = \{\Phi'(x)\}.$$

Proof. The strict curvature imposed by strong convexity eliminates facet degeneracies, ensuring uniqueness of supporting hyperplanes throughout the carrier interior. Consequently, strongly convex extensions possess globally well-defined gradient fields compatible with vertex interpolation constraints. ■

Differentiable convex extensions further refine this structure by guaranteeing gradient continuity. For $\Phi \in C^1$, on $\text{conv } E$ the gradient mapping $x \mapsto \Phi'(x)$ defines a continuous field encoding smooth interpolation propagation from vertex data into the carrier interior.

Higher smoothness classes yield higher-order geometric regularity, including curvature tensors and Hessian structures governing second-order extremal behavior.

The subdifferential geometry of convex extensions may therefore be summarized through the following hierarchy:

- minimal convex extensions — polyhedral subdifferential sets;
- strongly convex extensions — singleton interior subgradients
- smooth convex extensions — continuous gradient fields.

Each class reflects increasing analytic rigidity superimposed on the same vertex interpolation backbone.

The interaction between subdifferential structure and carrier normal cones forms the variational foundation for extremal analysis. Supporting hyperplanes, gradient growth, and optimality conditions are all governed by the geometry of $\partial\Phi(x)$ relative to $\text{conv } E$.

This variational structure provides the analytical apparatus required for establishing extremal bounds and optimization properties of convex

The structural analysis developed in the preceding blocks reveals that convex extensions on vertex-located configurations are governed by a unified geometric framework determined jointly by carrier polyhedrality, vertex extremality, and convex interpolation mechanisms.

Extension structure is therefore not arbitrary but is rigidly constrained by the geometry of the configuration hull.

A central conclusion of the structural theory is the principle of carrier determinacy: admissible convex extensions are fully confined within geometric and variational bounds imposed by $\text{conv } E$.

This determinacy manifests simultaneously in three structural layers.

First, epigraph geometry is restricted by the convex hull of lifted vertex data. No admissible convex extension may possess epigraph facets violating the convex envelope generated by configuration points.

Second, interpolation bounds constrain extension values at every carrier location. Convex combinations of vertex data generate admissibility envelopes within which all extensions must lie.

Third, subdifferential structure is governed by carrier normal cones and supporting hyperplanes induced by polyhedral extremality.

These three layers, namely epigraph structure, interpolation bounds, and subgradient geometry, form a mutually consistent structural system encoding continuation admissibility.

Each convex extension compatible with prescribed vertex data must satisfy all three structural constraints simultaneously.

The extremal rigidity of vertex-located configurations ensures that this structural system is finitely generated and combinatorially stratified. Carrier facets, vertex supports, and normal cones together determine admissible extension geometry throughout $\text{conv } E$.

Thus, discrete configuration extremality propagates into continuous analytic structure via convex continuation mechanisms.

A comparative view of extension classes reveals how analytic regularity modifies structural realizations without altering admissibility constraints.

Minimal convex extensions realize admissible bounds in polyhedral form. Their epigraphs are facet-stratified, their interpolation values attain extremal envelopes, and their subgradients form polyhedral cones.

Strongly convex extensions preserve admissibility constraints while introducing curvature. Their epigraphs become strictly convex, interpolation values separate quadratically from affine bounds, and subgradients become unique in interior regions.

Differentiable convex extensions refine this structure further by smoothing epigraph stratifications and generating continuous gradient fields compatible with carrier geometry.

Despite analytic differences, all extension classes remain governed by the same carrier-determined admissibility system.

Thus, analytic regularization modifies realization but not structural feasibility of convex extension.

The structural synthesis developed in this section provides the geometric and variational foundation for extremal analysis of convex extensions.

Since admissible extension values, gradients, and supports are constrained by carrier geometry, extremal properties of convex extensions must be studied relative to these structural bounds.

In particular, one may investigate:

- minimal achievable extension values;
- maximal admissible growth rates;
- curvature-induced extremal separation;
- gradient bounds imposed by carrier normals.

Such questions lead directly to the extremal theory of convex extensions, where functional inequalities, support characterizations, and optimality conditions are derived from structural constraints established above.

The transition from structural geometry to extremal functional analysis is therefore natural and methodologically continuous.

Convex extension constructions also admit a dual variational interpretation. While primal extension operators propagate interpolation values through convex envelopes, dual representations propagate admissible subgradient domains determined by carrier geometry. This dual perspective becomes particularly significant when analyzing extremal bounds and optimality conditions of convex extensions, which will be investigated in the following section.

This completes the structural study of convex extensions on vertex-located configurations.

In the following section, extremal properties of convex extensions will be investigated, including lower bounds, growth estimates, and sufficient optimality conditions determined by carrier extremality and interpolation structure.

§2.4. Extremal Properties of Convex Extensions on Vertex-Located Configurations

Having established the existence of convex extensions and analyzed their structural properties, we now turn to their extremal analysis. The objective is to derive lower bounds for the discrete minimum of a function f prescribed on a finite vertex-located configuration E , and to formulate sufficient conditions for a point of E to be a global minimizer. These results connect the discrete optimization problem on E with the analytic properties of a convex extension defined on the carrier polyhedron.

Throughout this section, $E = \{x^1, \dots, x^m\} \subset \mathbb{R}^n$ denotes a finite vertex-located configuration, $\Pi = \text{conv } E$ its carrier polyhedron, and $f: E \rightarrow \mathbb{R}$ a prescribed function. A convex extension $\varphi: \Pi \rightarrow \mathbb{R}$ satisfies the interpolation condition $\varphi(x) = f(x)$ for every $x \in E$. A fixed carrier point is denoted $\bar{x} \in \Pi$.

Three classes of lower bounds will be established, corresponding to the three extension classes of §2.2: a subgradient bound for general convex extensions, a gradient bound for differentiable extensions, and a quadratic bound for strongly convex extensions. Each bound sharpens its predecessor by incorporating additional regularity of the extension.

Theorem 2.13. (*Lower Bound via Convex Extension*).

Let φ be a convex extension of f from E to Π . Then for any point $\bar{x} \in \Pi$ and any subgradient $g \in \partial\varphi(\bar{x})$,

$$\min_{x \in E} f(x) \geq \varphi(\bar{x}) + \min_{x \in E} \langle g, x - \bar{x} \rangle. \quad (2.3)$$

Proof. Since φ is convex on Π , it satisfies the subgradient inequality: for every $\bar{x} \in \Pi$ and $g \in \partial\varphi(\bar{x})$,

$$\varphi(x) \geq \varphi(\bar{x}) + \langle g, x - \bar{x} \rangle \quad \text{for all } x \in \Pi.$$

This inequality asserts that the affine function

$$\ell(x) = \varphi(\bar{x}) + \langle g, x - \bar{x} \rangle$$

is a global affine minorant of φ on Π , representing a supporting hyperplane to the epigraph of φ at the point $(\bar{x}, \varphi(\bar{x}))$.

We now restrict x to the vertex set E . Since $\varphi(x) = f(x)$ for every $x \in E$ by the interpolation condition, the subgradient inequality yields

$$f(x) \geq \varphi(\bar{x}) + \langle g, x - \bar{x} \rangle \quad \text{for all } x \in E.$$

Since E is a finite set, we may take the minimum of both sides over $x \in E$. The left side gives

$$\min_{x \in E} f(x),$$

while the right side gives

$$\varphi(\bar{x}) + \min_{x \in E} \langle g, x - \bar{x} \rangle,$$

which completes the proof. ■

Remark 2.5. The estimate of Theorem 2.13 is purely geometric: it requires only convexity of the extension and the interpolation condition, and imposes no smoothness assumptions. For a fixed choice of $\bar{x}$ and g , the lower bound is determined by the minimum of a linear function over the finite set E , which is attained at a vertex of the configuration. The sharpness of the bound depends on how well the affine minorant ℓ approximates f near its minimum: the bound becomes tight precisely when the supporting hyperplane to $\text{epi } \varphi$ at $\bar{x}$ touches the graph of f at the minimizer.

The lower bound of Theorem 2.13 yields a sufficient condition for optimality on the finite configuration. A point $x^* \in E$ is a minimizer of f on E if and only if there exist $\bar{x} \in \Pi$ and $g \in \partial\varphi(\bar{x})$ such that

$$\langle g, x - \bar{x} \rangle \geq \langle g, x^* - \bar{x} \rangle \quad \text{for all } x \in E,$$

and the lower bound (2.3) is attained with equality at x^* . Geometrically, the affine minorant ℓ achieves its minimum over E precisely at x^* , so that x^* is simultaneously a vertex of E and a minimizer of the affine minorant of φ .

The bound of Theorem 2.13 may be refined when the convex extension is differentiable, since in this case the subdifferential reduces to a singleton and the supporting hyperplane at each carrier point is uniquely determined.

Theorem 2.14. (*Lower Bound via Differentiable Convex Extension*).

Let φ be a differentiable convex extension of f on Π . Then for any point $\bar{x} \in \Pi$,

$$\min_{x \in E} f(x) \geq \varphi(\bar{x}) + \min_{x \in E} \langle \varphi'(\bar{x}), x - \bar{x} \rangle. \quad (2.4)$$

Proof. Since φ is convex and differentiable on Π , its subdifferential at every point $\bar{x} \in \Pi$ consists of the single element $\varphi'(\bar{x})$. The first-order convexity condition therefore takes the form

$$\varphi(x) \geq \varphi(\bar{x}) + \langle \varphi'(\bar{x}), x - \bar{x} \rangle \quad \text{for all } x \in \Pi,$$

which asserts that the graph of φ lies above every tangent hyperplane.

Restricting to the vertex set E and applying the interpolation condition $\varphi(x) = f(x)$ for $x \in E$ yields

$$f(x) \geq \varphi(\bar{x}) + \langle \varphi'(\bar{x}), x - \bar{x} \rangle \quad \text{for all } x \in E.$$

Taking the minimum over $x \in E$ on both sides yields the stated bound. ■

Remark 2.6. The differentiable bound of Theorem 2.14 possesses a more explicit analytic structure than the subgradient bound of Theorem 2.13, since the gradient $\varphi'(\bar{x})$ is uniquely defined and varies continuously with $\bar{x}$. This makes the bound amenable to analytic evaluation and optimization over the choice of carrier point $\bar{x}$. In particular, the sharpest lower bound of the form (2.4) is obtained by solving the auxiliary problem of maximizing

$$\varphi(\bar{x}) + \min_{x \in E} \langle \varphi'(\bar{x}), x - \bar{x} \rangle \quad \text{over } \bar{x} \in \Pi,$$

which is a continuous optimization problem on the convex carrier domain.

The sufficient optimality condition admits the same structure as in the convex case: $x^* \in E$ is a minimizer of f if and only if there exists $\bar{x} \in \Pi$ such that

$$\langle \varphi'(\bar{x}), x - \bar{x} \rangle \geq \langle \varphi'(\bar{x}), x^* - \bar{x} \rangle \quad \text{for all } x \in E$$

and equality holds at $x = x^*$.

A further refinement is available when the extension is strongly convex, since strong convexity introduces a quadratic growth term that separates values at distinct points of the carrier domain.

Theorem 2.15. (*Lower Bound via Strongly Convex Extension*).

Let ψ be a differentiable strongly convex extension of f on Π with strong convexity parameter $\rho > 0$. Then for any point $\bar{x} \in \Pi$,

$$\min_{x \in E} f(x) \geq \psi(\bar{x}) + \min_{x \in E} \left\{ \langle \psi'(\bar{x}), x - \bar{x} \rangle + (\rho/2) \|x - \bar{x}\|^2 \right\}. \quad (2.5)$$

Proof. Strong convexity of ψ with parameter $\rho > 0$ means that for all $x, y \in \Pi$,

$$\psi(x) \geq \psi(y) + \langle \psi'(y), x - y \rangle + (\rho/2) \|x - y\|^2.$$

This inequality strengthens the first-order condition by adding a quadratic term that measures the uniform separation between the function and its tangent hyperplane at any carrier point. The graph of ψ lies strictly above every tangent hyperplane except at the point of tangency.

We set $y = \bar{x} \in \Pi$ and restrict x to the vertex set E . Applying the interpolation condition $\psi(x) = f(x)$ for $x \in E$ gives

$$f(x) \geq \psi(\bar{x}) + \langle \psi'(\bar{x}), x - \bar{x} \rangle + (\rho/2) \|x - \bar{x}\|^2 \text{ for all } x \in E.$$

Taking the minimum over $x \in E$ on both sides yields the stated bound. ■

Remark 2.7. The strong convexity parameter ρ is determined by the construction of the extension rather than by the configuration alone. When strongly convex extensions are obtained via the spherical projection procedure of Theorem 2.3, ρ reflects the curvature of the auxiliary function h . The quadratic term $(\rho/2) \|x - \bar{x}\|^2$ yields additional separation between vertex values and the lower bound, and this separation grows with the dispersion of the configuration E relative to the reference point $\bar{x}$. Consequently, the bound of Theorem 2.15 is most effective when $\bar{x}$ is chosen near the minimizer and the configuration vertices are well separated. Interior carrier points generally produce sharper bounds than boundary points, since the quadratic term is more pronounced away from supporting facets of Π .

Remark 2.8. The three lower bounds form a natural hierarchy governed by extension regularity. The subgradient bound (2.3) requires only convexity and applies universally. The gradient bound (2.4) requires differentiability and yields a more explicit and continuous estimate. The quadratic bound (2.5) requires strong convexity and provides the sharpest separation. Each successive bound refines its predecessor by incorporating one additional layer of analytic regularity, and the gain in sharpness is directly controlled by the corresponding regularity parameter.

The sufficient optimality conditions associated with Theorems 2.13 through 2.15 admit a unified geometric formulation. For a differentiable convex extension φ , a point $x^* \in E$ is a minimizer of f on E if and only if

$$\varphi'(\bar{x}) \in N_u(x^*)$$

for some $\bar{x} \in \Pi$, where $N_u(x^*)$ denotes the normal cone of the polyhedron Π at the vertex x^* . This condition asserts that the gradient of the extension at the

carrier point $\bar{x}$ must point in an outward normal direction of Π at the minimizing vertex: the tangent hyperplane to the graph of φ at $\bar{x}$ supports the carrier polyhedron at x^* from the outside. For strongly convex extensions, the normal cone condition remains necessary and sufficient, and the quadratic curvature of ψ ensures that the minimizer is isolated whenever the normal cone at x^* is a proper cone.

These extremal bounds and optimality conditions provide the analytic foundation for optimization constructions on combinatorial point configurations developed in the subsequent chapters. The lower bound estimates translate analytic properties of convex extensions into computable certificates for discrete optimality, while the normal cone characterization reveals the geometric structure of optimality in terms of the carrier polyhedron.

§2.5. Convex Extensions of Functions on a Hypersphere

The theory of convex extensions developed in §§2.2–2.4 rests on the notion of a vertex-located configuration whose natural carrier is the polyhedron $\Pi = \text{conv } E$. The present section addresses a qualitatively different geometric setting, arising when the prescribed data are supported on a hypersphere rather than on the vertices of a polytope. Convex extensions for functions prescribed on spherically located configurations require a different construction, since the hypersphere is not a convex set: the extension must be defined on a convex domain containing it, and convexity must be compatible with the interpolation condition on the sphere.

The construction resolves this problem through a penalty-based approach. A continuous function f prescribed on the hypersphere

$$S_r = \{x \in \mathbb{R}^n: \|x\| = r\}$$

is first extended to the closed ball

$$B_r = \{x \in \mathbb{R}^n: \|x\| \leq r\}$$

via radial projection, and a quadratic penalty term is then added to enforce convexity.

This approach parallels the strongly convex regularization of Theorem 2.3, though it operates in the reversed direction: whereas in §2.2 a quadratic term was added to an already convex extension to increase curvature, here the quadratic penalty is needed to create convexity where none is guaranteed by the radial extension alone.

Let $f: S_r \rightarrow \mathbb{R}$ be a continuous function. A finite subset

$$E = \{x^1, \dots, x^m\} \subset S_r$$

will be called a spherically located configuration. It arises naturally as the support of discrete data sampled from f on the sphere. Since S_r is not convex, convex extensions of f must be defined relative to a convex carrier domain containing S_r . The canonical choice is the closed ball B_r , which is convex and compact and contains S_r as its boundary.

Let $f: S_r \rightarrow \mathbb{R}$ be a prescribed function and let $D \subset \mathbb{R}^n$ be a convex domain satisfying $S_r \subset D$.

Definition 2.4. A function $F: D \rightarrow \mathbb{R}$ is called a convex extension of f on S_r if F is convex on D and $F(x) = f(x)$ for every $x \in S_r$.

The set D is called a convex carrier domain for the extension; the canonical choice is $D = B_r$. There are in general many such extensions, differing in their values and analytic properties in the interior of the carrier.

Lemma 2.1. (*Radial Extension*).

Let $f: S_r \rightarrow \mathbb{R}$ be a continuous function. The function $\tilde{f}: B_r \rightarrow \mathbb{R}$ defined by

$$\tilde{f}(x) = f(rx/\|x\|) \text{ for } x \in B_r \setminus \{0\},$$

and $\tilde{f}(0) = f(x^0)$, where $x^0 \in S_r$ is an arbitrary fixed point, is continuous on $B_r \setminus \{0\}$ and satisfies $\tilde{f}(x) = f(x)$ for every $x \in S_r$. The value $\tilde{f}(0)$ does not affect continuity on $B_r \setminus \{0\}$ and plays no role in the subsequent arguments.

Proof. The radial projection

$$\pi: B_r \setminus \{0\} \rightarrow S_r$$

given by

$$\pi(x) = rx/\|x\|,$$

is continuous on $B_r \setminus \{0\}$ since $\|x\| > 0$ there and the map $x \mapsto x/\|x\|$ is continuous away from the origin. Since f is continuous on the compact set S_r , the composition $\tilde{f} = f \circ \pi$ is continuous on $B_r \setminus \{0\}$. For any $x \in S_r$ one has $\|x\| = r$. So $\pi(x) = x$ and hence $\tilde{f}(x) = f(x)$. ■

The radial extension $\tilde{f}$ is generally not convex on B_r ; the radial projection π distorts distances and can introduce negative curvature into $\tilde{f}$ even when f is smooth on S_r . This failure of convexity motivates the penalty construction that follows.

With $\tilde{f}$ in hand, define the penalty extension

$$F_p(x) = \tilde{f}(x) + p(\|x\|^2 - r^2), \quad x \in B_r, \quad (2.6)$$

where $p > 0$ is a penalty parameter.

The term $p(\|x\|^2 - r^2)$ is non-positive throughout B_r and vanishes exactly on S_r ; it contributes a constant positive curvature $2pI$ to the Hessian of F_p , which will be used to compensate the negative curvature of $\tilde{f}$.

Theorem 2.16. (*Existence of Convex Extension on a Hypersphere*).

Let $f: S_r \rightarrow \mathbb{R}$ be a continuous function. Then there exists a threshold $p^* > 0$ such that for every $p \geq p^*$ the function F_p defined by (2.6) is convex on B_r and satisfies $F_p(x) = f(x)$ for every $x \in S_r$.

Proof. The interpolation condition is immediate: for any $x \in S_r$ one has $\|x\|^2 = r^2$, so the penalty term vanishes and $F_p(x) = \tilde{f}(x) = f(x)$. This holds for any $p > 0$, so it remains only to establish convexity for p sufficiently large.

To analyze convexity, we work on the punctured ball $B_r \setminus \{0\}$, where the radial projection π is smooth. Assuming $f \in C^2(S_r)$, the composition $\tilde{f} = f \circ \pi$ is $C^2(B_r \setminus \{0\})$ as a composition of smooth maps, and the Hessian of F_p satisfies

$$F_p''(x) = \tilde{f}''(x) + 2pI$$

throughout $B_r \setminus \{0\}$.

Since $B_r \setminus \{0\}$ is the union of compact annuli $\{x : \delta \leq \|x\| \leq r\}$ for $\delta > 0$, and $\tilde{f}''$ is continuous on each such compact set, its smallest eigenvalue attains its minimum on each annulus.

The curvature of $\tilde{f}$ on any annulus is determined entirely by the curvature of f on the compact sphere S_r and the smooth structure of π on that annulus; it does not degenerate as $\delta \rightarrow 0$, since the Hessian of $\tilde{f} = f \circ \pi$ is controlled uniformly by the bounded derivatives of f on S_r . Consequently there exists a finite constant $M \geq 0$, depending only on f and r , such that

$$\lambda_{\min}(F_p''(x)) \geq -M \text{ for all } x \in B_r \setminus \{0\}.$$

Setting $p^* = M/2$, for every $p \geq p^*$ and all $x \in B_r \setminus \{0\}$ one obtains

$$\lambda_{\min}(F_p''(x)) \geq -M + 2p \geq -M + M = 0,$$

so $F_p''(x)$ is positive semi-definite throughout $B_r \setminus \{0\}$, which implies convexity of F_p on this open set. Since F_p is continuous on the compact set B_r and convex on the dense open subset $B_r \setminus \{0\}$, it is convex on all of B_r by a standard continuity argument. Moreover, for $p \geq p^*$ the Hessian is uniformly positive definite on compact subsets of $B_r \setminus \{0\}$, so F_p is strongly convex on B_r with modulus $2p - M$. ■

The threshold p^* reflects the worst-case negative curvature of the radial extension $\tilde{f}$: the more negative the curvature of $\tilde{f}$ on B_r , the larger the penalty required to restore convexity. For $p \geq p^*$ the extension F_p is strongly convex with modulus $2p - M$, and differentiable on the interior of B_r with a unique gradient at every interior point.

When a finite spherically located configuration $E \subset S_r$ is specified, the lower bound estimates of §2.4 apply directly to F_p on B_r . Since F_p is a strongly convex differentiable extension satisfying

$$F_p(x) = f(x)$$

for all $x \in E$, Theorems 2.14 and 2.15 of §2.4 yield a gradient lower bound and a quadratic lower bound for the discrete $\min\{f(x) : x \in E\}$, and the sufficient optimality condition is expressed via the normal cone of B_r at the minimizer.

Remark 2.9. A configuration E is polyhedrally-spherical if it is simultaneously vertex-located and spherically located:

$$E = \text{vert conv } E \cap S_r.$$

For such configurations, both the polyhedral construction of §2.2 and the penalty construction of the present section are applicable. The polyhedral approach constructs extensions on the carrier polyhedron $\Pi = \text{conv } E$, exploiting the vertex structure of E ; the penalty approach constructs extensions on the ball B_r , exploiting the spherical constraint. Both satisfy the interpolation condition on E , but extend f to different carrier domains with different analytic properties.

For optimization problems where the polyhedral geometry of Π is explicitly known, the polyhedral extension and the lower bounds of §2.4 are typically more tractable; when the spherical constraint is the dominant geometric feature, as in problems involving unit vectors or normalized data, the penalty extension F_p is more natural. The choice between the two approaches is determined by the geometric structure of the problem.

The results of §§2.1–2.5 complete the existence and structural theory of convex extensions for the three configuration classes introduced in §2.1. Vertex-located configurations admit polyhedral, strongly convex, and differentiable extensions on the carrier polyhedron, as established in §§2.2–2.3 and analyzed in §2.4. Spherically located configurations admit strongly convex differentiable extensions on the ball via the penalty construction of the present section. For polyhedrally-spherical configurations both types of extensions are available, and the choice between them reflects the dominant geometric structure of the problem.

The theoretical framework developed in this chapter, comprising existence theorems, structural properties, and extremal estimates, constitutes the analytic core of the monograph. Its application requires that the abstract class of vertex-located configurations be populated with concrete geometric objects, a task undertaken in Chapter 3, which introduces the principal combinatorial configuration classes, establishes their vertex-located and polyhedrally-spherical character, and determines the explicit polyhedral geometry of their convex hulls.

Chapter 3

Combinatorial Point Configurations and Their Polyhedral Geometry

The general theory of convex extensions developed in Chapter 2 was formulated at an abstract geometric level, independently of the specific origin of the point configurations involved. Concrete applications of this theory require explicit classes of vertex-located configurations whose polyhedral geometry is sufficiently well understood to make the extension constructions tractable. This chapter provides such realizations by introducing and systematically studying five hierarchically related classes of combinatorial point configurations generated by ordered mappings.

The construction rests on two primitive operations: a coordinate realization that assigns numerical values to abstract combinatorial elements, and an immersion that transfers families of ordered sequences into point sets in $\mathbb{R}^n$ (§3.1). This framework establishes a systematic bridge between discrete combinatorial structures and finite geometric domains amenable to convex and polyhedral analysis. Permutation and multipermutation point configurations are introduced in §3.2 as the full-occupancy regime, in which every admissible sequence uses all elements of the coordinate realization with prescribed multiplicities. Both classes are unconditionally vertex-located and admit polyhedral spherical representations. Their convex hulls, the permutohedron and multipermutohedra, are completely characterized through the face partition correspondence and explicit f -vector formulas in §3.3.

Partial multipermutation point configurations extend the full-occupancy model by allowing bounded repetitions together with partial coordinate selection (§3.4). Their convex hulls, the partial multipermutohedra, are studied in §3.5, where vertex criteria, adjacency structures, and halfspace representations are established. Partial permutation point configurations constitute the injective boundary of this class, and their vertex-located character is characterized by an explicit contiguous-block condition (§3.6). The chapter concludes with unbounded configurations and their Boolean specialization, whose convex hulls are hyperrectangles and which constitute the maximally regular members of the hierarchy (§3.7).

Together, the configuration classes introduced in this chapter populate the abstract framework of Chapter 2 with concrete geometric objects. The vertex-located and polyhedral spherical properties established here make the full apparatus of convex extension theory directly applicable to the combinatorial optimization problems studied in Chapters 4 and 5.

§3.1. Ordered Combinatorial Point Configurations and Their Euclidean Immersion

Classical combinatorics deals with finite collections of abstract objects defined through admissibility rules and structural constraints. Permutations, arrangements, combinations, and multisets each constitute a combinatorial class governed by its own structural rules: whether repetitions are admissible, whether order is significant, how many elements may be selected. In their abstract form, such objects carry no metric or affine structure. A permutation is not inherently a vector, and a combinatorial class is not inherently a point set. The present chapter effects the transition from this abstract combinatorial level to a geometric one, realizing families of ordered combinatorial objects as finite point sets in Euclidean space and studying their convex and polyhedral properties.

The study is restricted to ordered combinatorial objects: permutations and arrangements, with and without repetitions, together with their special cases. This restriction is motivated by three structural features that make such objects particularly suitable for geometric realization. Each object is represented as an ordered sequence of fixed length n ; positional information is intrinsic and cannot be discarded without destroying the object's structure; and the length n is known in advance, determining the ambient space $\mathbb{R}^n$. These properties are absent in unordered objects such as combinations or set partitions, whose geometric realizations, when they exist, require fundamentally different constructions and fall outside the scope of this monograph.

The combinatorial background for admissible mappings is provided by the framework introduced by Berge [18], in which a *combinatorial configuration* is specified by a feasible mapping

$$\chi: B \rightarrow A,$$

where B is a finite set of positions, A is an abstract resulting set, and χ satisfies a prescribed constraint system.

In the present study, the position set is the standard index set

$$J_n = \{1, \dots, n\},$$

and each feasible mapping defines a single ordered combinatorial object. This abstract framework provides terminological context; the working apparatus is built on numerical multisets, which simultaneously encode combinatorial structure and coordinate geometry.

The central object of the framework is a finite numerical multiset that serves as both combinatorial alphabet and coordinate carrier. Let

$$G = \{g_1^{n_1}, \dots, g_k^{n_k}\}$$

be a finite multiset of real numbers, where $g_1 < \dots < g_k$ are pairwise distinct values and $n_1, n_2, \dots, n_k$ are positive integers specifying their multiplicities.

Three objects associated with G are used throughout the chapter.

The *underlying set* $S(G) = \{g_1, \dots, g_k\}$ consists of the distinct elements.

The *primary specification* $[G] = (n_1, \dots, n_k)$ records the multiplicities.

The *total multiplicity* $\eta = n_1 + \dots + n_k = |G|$ is the number of elements counted with repetitions.

The multiset G may also be described by its *ordered representation*

$$A = (a_1 \leq \dots \leq a_\eta),$$

in which each value g_i appears n_i times.

When $G = S(G)$, equivalently when all multiplicities equal one, the multiset reduces to a plain set and $\eta = k$.

Definition 3.1. The triple $(G, S(G), [G])$ consisting of the numerical multiset, its underlying set, and its primary specification is called the *coordinate realization*.

The parameters $k = |S(G)|$ and $\eta = |G|$ are the number of distinct coordinate values and the total multiplicity, respectively.

Once the coordinate realization G is fixed, the mechanism for transferring ordered combinatorial objects into Euclidean space is constructed as follows. An ordered combinatorial object of length n is an ordered selection of n elements from A subject to prescribed constraints. Such a selection is a sequence

$$\pi = (a_{i_1}, \dots, a_{i_n}), \quad i_1, \dots, i_n \in J_\eta,$$

where the index sequence $(i_1, \dots, i_n)$ satisfies the constraints of the combinatorial class under consideration.

Each such sequence defines a point in $\mathbb{R}^n$ by the coordinate assignment $x = (a_{i_1}, \dots, a_{i_n})$. This defines the *immersion*

$$\varphi : \pi \mapsto x = \varphi(\pi) \in \mathbb{R}^n.$$

Let Π denote the set of all admissible sequences of length n drawn from A under a fixed constraint system Λ . Applying φ to every element of Π yields the image set $E = \varphi(\Pi) \subset \mathbb{R}^n$.

Definition 3.2. The finite point set $E = \varphi(\Pi) \subset \mathbb{R}^n$ obtained as the image of the set of all admissible sequences under the immersion φ is called a *combinatorial*

point configuration (CPC) generated by the coordinate realization G and constraint system Λ .

A CPC is a collective object. A single sequence π corresponds to a single point $x \in E$ and by itself carries no geometric structure. Meaningful geometric properties such as convex hulls, affine dimension, symmetry, spherical localization, and polyhedral decompositions arise only at the level of the entire set E . All geometric analysis in this chapter is performed at this collective level.

Remark 3.1. The immersion φ depends on the coordinate realization G but not on the abstract nature of the underlying combinatorial objects. Affinely equivalent realizations, related by a linear transformation $g \mapsto \alpha g + \beta$ with $\alpha \neq 0$, produce affinely equivalent configurations with identical combinatorial structure. The canonical realization $G = \{1, \dots, n\}$ with $[G] = (1^n)$ is particularly convenient for explicit computations and is used in §3.3.

Each point $x \in E$ carries its own multiset of coordinate values, encoding the repetition structure of the corresponding sequence.

Definition 3.3. For a point $x \in E$, the multiset $G(x) = \{x^1, \dots, x_n\}$ with elements listed according to their multiplicities in the coordinate vector is called the *inducing multiset* of x . The multiset

$$G(E) = \bigcup_{x \in E} G(x)$$

is called the inducing multiset of the configuration E .

Throughout this chapter the condition $G(E) = G$ is assumed.

Three parameters govern the combinatorial class to which a configuration belongs: the sequence length n , the total multiplicity $\eta = |G|$, and the number of distinct values $k = |S(G)|$.

Five principal classes of CPCs arise by varying the constraints imposed on admissible sequences.

Definition 3.4. A CPC $E \subset \mathbb{R}^n$ is called a *permutation point configuration (PPC)* if $G(x) = S(G)$ for every $x \in E$ and $\eta = n = k$.

Every sequence uses each element of $S(G)$ exactly once, corresponding to classical permutations without repetitions.

The entire PPC is denoted E_n or $E_n(G)$, and its convex hull

$$\Pi_n = \text{conv } E_n$$

is the associated *permutohedron*.

Definition 3.5. A CPC $E \subset \mathbb{R}^n$ is called a *multi-permutation point configuration (MPPC)* if $G(x) = G$ for every $x \in E$ and $\eta = n \geq k$.

Every sequence uses all η elements of G counted with multiplicities.

When $\eta = k$ the MPPC reduces to a PPC.

The entire MPPC is denoted E_{nk} or $E_{nk}(G)$, and its convex hull

$$\Pi_{nk} = \text{conv } E_{nk}$$

is the associated *multipermutohedron*.

Definition 3.6. A CPC $E \subset \mathbb{R}^n$ is called a *partial permutation point configuration* (PPPC) if $G(x) = S(G(x)) \subset S(G)$ for every $x \in E$ and $n < \eta = k$.

No repetitions occur within individual points, but not all k values need be used in each sequence since $n < k$.

The entire PPPC is denoted E_n^k or $E_n^k(G)$, and its convex hull

$$\Pi_n^k = \text{conv } E_n^k$$

is the associated *partial permutohedron*.

Definition 3.7. A CPC $E \subset \mathbb{R}^n$ is called a *partial multipermutation point configuration* (PMPC) if $G(x) \subset G$ for every $x \in E$ and $n < \eta, k \leq \eta$.

This is the most general bounded class, allowing both partial occupancy and bounded repetitions.

A PMPC reduces to a PPPC when $\eta = k$.

The entire PMPC is denoted $\tilde{E}_n^k$ or $\tilde{E}_n^k(G)$, and its convex hull

$$\tilde{\Pi}_n^k = \text{conv } \tilde{E}_n^k$$

is the associated *partial multipermutohedron*.

Definition 3.8. A CPC $E \subset \mathbb{R}^n$ is called an *unbounded partial multipermutation point configuration* (UPMPC) if coordinates vary freely over $S(G)$ without multiplicity restrictions, so that

$$E = S(G)^n = B_n^k$$

and $n_i \geq n$ for all i .

The entire UPMPC is denoted $\tilde{E}_n^k$ or $\tilde{E}_n^k(G)$, and its convex hull

$$\Pi_n^\infty = \text{conv } B_n^k$$

is a hyperrectangle.

Definition 3.9. A CPC $E \subset \mathbb{R}^n$ is called a *Boolean point configuration* (BPC) if it is a UPMPC with two-element realization $S(G) = \{b_1, b_2\}$, $b_1 < b_2$.

The entire BPC is denoted $B_n^2 = \{b_1, b_2\}^n$, and its convex hull

$$\text{conv } B_n^2 = [b_1, b_2]^n$$

is the associated *Boolean hypercube*.

In the canonical realization B_n^2 the configuration coincides with the standard Boolean cube $[0, 1]^n$.

The five classes form a coherent hierarchy governed by two structural parameters: degree of occupancy and degree of repetition admissibility, summarized in Table 3.1.

Table 3.1. Classification of combinatorial point configurations by parameters n, η, k

Class	Configuration	Polytope	Parameters	Condition
PPC	$E_n(G)$	$\Pi_n(G)$	$\eta = n = k$	Full occupancy, no repetitions
MPPC	$E_{nk}(G)$	$\Pi_{nk}(G)$	$\eta = n \geq k$	Full occupancy, bounded repetitions
PPPC	$E_n^k(G)$	$\Pi_n^k(G)$	$n < \eta = k$	Partial occupancy, no repetitions
PMPC	$\tilde{E}_n^k(G)$	$\tilde{\Pi}_n^k(G)$	$n < \eta,$ $k \leq \eta$	Partial occupancy, bounded repetitions
UPMPC	$B_n^k(G)$	$\Pi_n^\infty(G)$	$\eta_i \geq n$	Partial occupancy, no bounds
Boolean	$B_n^2(G)$	$[b_1, b_2]^n$	$k = 2$	UPMPC with two-element realization

The immersion φ transfers not only individual combinatorial objects but their collective structure into Euclidean space. The resulting configurations inherit geometric properties depending on the constraint class and the coordinate realization G . Among these, vertex-locatedness plays a central role: a CPC E is *vertex-located* if $E = \text{vert conv } E$, meaning every point of E is a vertex of its convex hull.

§3.2. Permutation and Multipermutation Point Configurations

Permutation and multipermutation point configurations constitute the full-occupancy regime of the hierarchy introduced in §3.1: every admissible sequence of length n uses all η elements of the coordinate realization G with their prescribed multiplicities. This constraint produces configurations whose geometry is highly symmetric and structurally transparent. The present section establishes their fundamental geometric properties: affine localization, spherical localization, vertex-locatedness, polyhedral-spherical representation, hyperplane decompositions, and adjacency structure. These results serve as the geometric foundation for all subsequent sections.

Throughout this section the coordinate realization G has underlying set $S(G) = \{g_1 < \dots < g_k\}$, primary specification $[G] = (n_1, \dots, n_k)$, and total multiplicity $\eta = n_1 + \dots + n_k = n$. The ordered representation is $A = (a_1 \leq \dots \leq a_n)$, containing each g_i exactly n_i times. The condition $\eta = n$ distinguishes full occupancy from the partial regimes studied in §§3.4–3.6.

For convenience we set

$$S_j = \sum_{i=1}^n (a_i)^j, j \in \mathbb{N}.$$

The values S_1 and S_2 play a central role throughout:

$$S_1 = \sum_{i=1}^n a_i,$$

$$S_2 = \sum_{i=1}^n a_i^2.$$

The entire multipermutation point configuration $E_{nk}(G) = \varphi(P_{nk})$ consists of all Euclidean images of sequences that are rearrangements of A . Its cardinality is given by the multinomial coefficient

$$|E_{nk}(G)| = \frac{n!}{n_1! \dots n_k!}$$

When all multiplicities equal 1, so that $[G] = (1^n)$ and $\eta = k = n$, the configuration reduces to the entire permutation point configuration $E_n(G)$ of cardinality $n!$, consisting of all $n!$ permutations of the n distinct coordinate values.

Theorem 3.1. (*Affine localization of MPPC*).

Every point $x \in E_{nk}(G)$ satisfies the affine constraint

$$\sum_{i=1}^n x_i = S_1,$$

where

$$S_1 = \sum_{i=1}^n a_i = \sum_{j=1}^k n_j g_j. \quad (3.1)$$

Consequently, $E_{nk}(G)$ lies in the affine hyperplane

$$H = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i = S_1 \right\},$$

and $\dim E_{nk}(G) \leq n - 1$.

Proof. Every point $x \in E_{nk}(G)$ is a permutation of the ordered representation $A = (a_1, \dots, a_n)$. Permuting the coordinates does not change their sum, so

$$\sum_{i=1}^n x_i = \sum_{i=1}^n a_i = S_1$$

holds for every $x \in E_{nk}(G)$.

The hyperplane H is an $(n-1)$ -dimensional affine subspace of $\mathbb{R}^n$, and in generic cases the configuration $E_{nk}(G)$ is full-dimensional within H , giving $\dim E_{nk}(G) = n - 1$. ■

Theorem 3.2. (*Spherical localization of MPPC*).

Every point $x \in E_{nk}(G)$ satisfies

$$\sum_{i=1}^n x_i^2 = S_2,$$

where

$$S_2 = \sum_{i=1}^n a_i^2 = \sum_{j=1}^k n_j g_j^2. \quad (3.2)$$

Consequently, $E_{nk}(G)$ is spherically located: it lies on the hypersphere

$$S_r(\mathbf{0}) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 = S_2 \right\},$$

centered at the origin $\mathbf{0} \in \mathbb{R}^n$ with radius $r = \sqrt{S_2}$.

Proof. Permuting the coordinates of $(a_1, \dots, a_n)$ does not change the sum of squares, so

$$\sum_{i=1}^n x_i^2 = \sum_{i=1}^n a_i^2 = S_2$$

holds for every $x \in E_{nk}(G)$. ■

Theorem 3.3. (*Circumsphere family*). The configuration $E_{nk}(G)$ lies on the one-parameter family of hyperspheres, parameterized by $\alpha \in \mathbb{R}$, the family of hyperspheres:

$$S_{r(\alpha)}(\tau) = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n (x_i - \alpha)^2 = r(\alpha)^2 \right\} \quad (3.3),$$

centered at $\tau = (\alpha, \dots, \alpha) \in \mathbb{R}^n$ with radius

$$r(\alpha) = \sqrt{S_2 - 2\alpha S_1 + n\alpha^2}. \quad (3.4)$$

For the circumscribed sphere, the minimum radius holds:

$$r_{\min} = r(\alpha_{\min}) = \sqrt{S_2 - \frac{S_1^2}{n}},$$

where

$$\alpha_{\min} = \frac{S_1}{n}.$$

Proof. Expanding

$$\sum_{i=1}^n (x_i - \alpha)^2$$

and substituting (3.1) and (3.2) gives (3.4) for every point of $E_{nk}(G)$.

This is a quadratic in α with minimum at $\alpha_{min} = S_1/n$, from which r_{min} follows by substitution. ■

The center $\tau = (\alpha_{min}, \dots, \alpha_{min}) \in \mathbb{R}^n$ is the centroid of the configuration, and both the center and the radius are determined entirely by the moments S_1 and S_2 of the coordinate realization G .

Theorem 3.4. (*Vertex-locatedness of MPPC*). The entire MPPC is vertex-located:

$$vert \Pi_{nk}(G) = E_{nk}(G),$$

where $\Pi_{nk}(G) = conv E_{nk}(G)$.

Proof. By Theorem 3.2, $E_{nk}(G)$ is spherically located. By Theorem 1.1 of Chapter 1, every finite spherically located configuration is vertex-located. Therefore $E_{nk}(G) = vert \Pi_{nk}(G)$. ■

The polytope $\Pi_{nk}(G) = conv E_{nk}(G)$ is called the *multipermutohedron* associated with G .

In the nondegenerate case

$$[G] = (1^n),$$

when $\Pi_{nk}(G)$ reduces to $\Pi_n(G)$, it is the classical *permutohedron*.

Vertex-locatedness means that no point of $E_{nk}(G)$ can be removed without altering the polytope, making MPPCs natural domains for the convex extension constructions of Chapter 2: any function

$$f: E_{nk}(G) \rightarrow \mathbb{R}$$

can be extended to a convex function on $\Pi_{nk}(G)$, with the extension determined entirely by its values at the vertices.

Theorem 3.5. (*Polyhedral-spherical representation*).

For any $\alpha \in R$, the entire MPPC admits the representation

$$E_{nk}(G) = \Pi_{nk} \cap S_{r(\alpha)}(\alpha),$$

where $S_{r(\alpha)}(\alpha)$ is any sphere from the circumsphere family (3.3).

Proof. By Theorem 3.4,

$$E_{nk}(G) \subset \Pi_{nk}(G).$$

By Theorem 3.3,

$$E_{nk}(G) \subset S_{r(\alpha)}(\alpha)$$

for every α . Hence

$$E_{nk}(G) \subseteq \Pi_{nk} \bigcap S_{r(\alpha)}(\alpha).$$

Conversely, let

$$x \in \Pi_{nk} \bigcap S_{r(\alpha)}(\alpha).$$

Since x lies on the hypersphere $S_{r(\alpha)}(\alpha)$, which is a strictly convex surface, and x belongs to the convex hull

$$\Pi_{nk}(G) = \text{conv } E_{nk}(G),$$

Theorem 1.1 of Chapter 1 implies that x must be a vertex of $\Pi_{nk}(G)$.

By Theorem 3.4,

$$\text{vert } \Pi_{nk}(G) = E_{nk}(G),$$

so $x \in E_{nk}(G)$.

Therefore

$$\Pi_{nk} \bigcap S_{r(\alpha)}(\alpha) \subseteq E_{nk}(G),$$

and equality holds. ■

Beyond the global affine and spherical constraints, MPPCs possess a rich internal stratification into parallel hyperplane layers that encodes combinatorial information about the underlying permutation structure.

Theorem 3.6. (*Hyperplane decomposition*).

For every subset $\omega \subset J_n$ with $|\omega| = s$, the configuration $E_{nk}(G)$ decomposes into the disjoint union

$$E_{nk}(G) = \bigcup_{t \in T_{G,s}} E^{t\omega},$$

where

$$E^{t\omega} = E_{nk}(G) \bigcap \{x : \sum_{i \in \omega} x_i = t\},$$

and $T_{G,s}$ is the set of all values attainable by

$$\sum_{i \in \omega} x_i$$

for $x \in E_{nk}(G)$.

Proof. Each point $x \in E_{nk}(G)$ has a unique value of

$$\sum_{i \in \omega} x_i$$

determined by which elements of G occupy the positions indexed by ω . This value lies in $T_{G,s}$ by definition, the layers are disjoint, and their union covers $E_{nk}(G)$. ■

The extreme layers, corresponding to the minimum and maximum values in $T_{G,s}$ are related to supporting hyperplanes of $\Pi_{nk}(G)$ and correspond to its faces. These hyperplane decompositions play a central role in the geometric analysis of §3.3 and provide the structural basis for constructing convex extensions of functions on multipermutation domains.

The edge structure of the multipermutohedron admits a precise combinatorial characterization. Two vertices $x, y \in E_{nk}(G)$ are *adjacent* in $\Pi_{nk}(G)$ if and only if they are connected by an edge of the polytope.

Theorem 3.7. (*Adjacency criterion*).

Two distinct points $x, y \in E_{nk}(G)$ are adjacent in $\Pi_{nk}(G)$ if and only if y is obtained from x by transposing two coordinates $x_i = g_p$ and $x_j = g_q$ with $g_p \neq g_q$, and no value g_h with $g_p < g_h < g_q$ (or $g_q < g_h < g_p$) is available for simultaneous transfer in the opposite direction. The degree of every vertex x in the adjacency graph of $\Pi_{nk}(G)$ equals

$$R(x) = \sum_{\substack{1 \leq p < q \leq k \\ g_p \neq g_q}} n_p n_q$$

In the nondegenerate case $[G] = (1^n)$, every vertex has degree

$$R = \frac{n(n-1)}{2},$$

confirming that the classical permutohedron is a simple polytope with every vertex on exactly $n - 1$ edges.

Remark 3.2. The geometric properties established in Theorems 3.1–3.7 depend only on the coordinate realization G and not on the abstract nature of the underlying combinatorial objects. Affinely equivalent realizations, related by $g \mapsto \alpha g + \beta$ with $\alpha \neq 0$, produce affinely equivalent configurations and multipermutohedra with identical combinatorial structure. The moments S_1 and S_2 transform as $S_1 \mapsto \alpha S_1 + n\beta$ and $S_2 \mapsto \alpha^2 S_2 + 2\alpha\beta S_1 + n\beta^2$, so the circumsphere center and radius scale accordingly. In particular, all vertex-locatedness, polyhedral-spherical, and adjacency results are invariant under affine equivalence.

The results of this section establish the complete geometric framework for permutation and multipermutation point configurations. In the nondegenerate case $[G] = (1^n)$, the permutation point configuration $E_n(G)$ generates the classical permutohedron $\Pi_n(G)$, whose affine localization, spherical structure, adjacency relations, and face lattice admit an especially transparent description. This canonical setting is analyzed in full detail in §3.3. When coordinate multiplicities are introduced, the permutation structure undergoes systematic modifications encoded in the primary specification $[G]$, giving rise to the generalized multipermutohedra $\Pi_{nk}(G)$. The partial occupancy regime, in which sequences use only a proper subset of the available coordinate values, is studied in §§3.4–3.6.

§3.3. Geometry of the Permutohedron and Ordered Partitions

The results of §3.2 apply to multipermutation point configurations under an arbitrary coordinate realization G . The present section specializes to the canonical realization $G = J_n = \{1, \dots, n\}$ with $[G] = (1^n = (1, \dots, 1))$ and $\eta = k = n$, which corresponds to the nondegenerate PPC regime $\eta = n = k$.

In this setting all structural properties of the permutation point configuration and its convex hull attain their most transparent and complete description. The entire PPC is denoted E_n and the associated permutohedron $\Pi_n = \text{conv } E_n$. All results of §3.2 apply with $n_i = 1$ for all i , giving

$$S_1 = \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$S_2 = \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

By Theorem 3.1, E_n lies in the affine hyperplane

$$\sum_{i=1}^n x_i = \frac{n(n+1)}{2}, \quad (3.5)$$

and Π_n is a polytope of dimension $n - 1$ contained in this hyperplane.

By Theorems 3.2 and 3.3, the minimum circumsphere of E_n has radius

$$r_{\min} = \sqrt{S_2 - \frac{S_1^2}{n}} = \sqrt{\frac{n(n+1)(n-1)}{12}}$$

and center

$$\tau = (\alpha, \dots, \alpha) = \left(\frac{n+1}{2}, \dots, \frac{n+1}{2} \right)$$

By Theorem 3.4, every point of E_n is a vertex of Π_n , and by Theorem 3.5 the polyhedral-spherical representation

$$E_n = \Pi_n \cap \bigcap S_{r(\alpha)}(\alpha)$$

holds for every sphere from the circumsphere family (3.3).

The irredundant H-representation of Π_n is given by the affine equation (3.5) together with the system of inequalities

$$\sum_{j \in \omega} x_j \geq \frac{s(s+1)}{2}, \quad (3.6)$$

where $\omega \subset J_n$, $|\omega| = s$, $1 \leq s \leq n-1$.

Each of the $2^n - 2$ inequalities in (3.6) is facet-defining. Together with the affine constraint (3.9) they completely determine Π_n as a convex polytope of dimension $n-1$.

The edge structure of Π_n follows from Theorem 3.7 with $k = n$ and $[G] = (1^n)$. Since all multiplicities equal one, the degree formula (3.8) gives $R = n-1$, confirming that Π_n is a *simple* polytope: every vertex lies on exactly $n-1$ edges. The total number of edges is

$$|\text{edges}(\Pi_n)| = \frac{(n-1)n!}{2}.$$

All edges of Π_n are parallel to vectors of the form $e_i - e_j$, where e_i and e_j are canonical basis vectors in $\mathbb{R}^n$. By Theorem 3.7 applied to the canonical realization, the exchanged values must be consecutive integers m and $m+1$ for some $m \in \{1, \dots, n-1\}$, so the 1-skeleton of Π_n is the Cayley graph of the symmetric group S_n generated by adjacent transpositions.

By Theorem 3.6, for any subset $\omega \subset J_n$ with $|\omega| = s$ the configuration E_n decomposes into exactly C_n^s non-empty layers. Each layer E_n^ω consists of all permutations in which the values occupying the positions indexed by ω sum to t . The extreme layers, corresponding to

$$t = \frac{s(s+1)}{2}$$

and

$$t = \frac{s(2n-s+1)}{2},$$

are vertex sets of faces of Π_n and are each in bijection with a direct product $E_s \times E_{n-s}$ of two smaller permutation point configurations.

The complete face structure of Π_n is governed by a fundamental combinatorial correspondence. Recall that an *ordered partition* of J_n is a sequence of nonempty pairwise disjoint subsets $(B_1, \dots, B_r)$ whose union is J_n . The number r of blocks is the *type* of the partition. One partition is a *refinement* of another if each block of the coarser partition is a union of blocks of the finer one.

Theorem 3.8. (*Face-partition correspondence*).

There is a bijection between the faces of Π_n and the ordered partitions of J_n . Under this bijection a face F of dimension d corresponds to an ordered partition

of J_n into $n - d$ blocks. In particular, vertices correspond to ordered partitions into n singleton blocks, edges correspond to partitions containing exactly one block of size two, facets correspond to partitions into exactly two blocks, and Π_n itself corresponds to the trivial one-block partition.

Proof. A face F of Π_n arises when a compatible family of the inequalities (3.6) becomes simultaneously active. Compatibility means the corresponding index sets form a chain

$$\omega_1 \subset \cdots \subset \omega_m \subset J_n.$$

Each such chain decomposes J_n into $m + 1$ successive difference blocks

$$B_1 = \omega_1, B_2 = \omega_2 \setminus \omega_1, \dots, B_{m+1} = J_n \setminus \omega_m.$$

Within each block B_r the coordinates carry the same relative range of values and may be permuted freely among the block positions; the relative order between blocks is fixed.

The dimension of F equals the total number of free transpositions, which is

$$\sum_r (|B_r| - 1) = n - (m + 1).$$

Setting $r = m + 1$ gives $\dim F = n - r$. The one-to-one correspondence between chains of subsets of J_n and ordered partitions of J_n establishes the bijection. ■

The geometric content of this correspondence is transparent. An ordered partition $(B_1, \dots, B_r)$ specifies a partial ordering of the coordinate positions: positions in B_1 carry the $|B_1|$ smallest coordinate values, positions in B_2 carry the next $|B_2|$ smallest values, and so on. Within each block the values may be distributed freely, while the relative order between blocks is fixed. The corresponding face is itself a permutohedron of dimension

$$|B_1| - 1 + \cdots + |B_r| - 1 = n - r$$

in its own affine hull.

Remark 3.3. The face lattice of Π_n , ordered by inclusion, is anti-isomorphic to the refinement lattice of ordered partitions of J_n . Refining a partition (splitting blocks) corresponds to passing to a lower-dimensional face, whereas coarsening (merging consecutive blocks) corresponds to passing to a higher-dimensional face.

Theorem 3.9. (*f-vector of the permutohedron*).

The permutohedron Π_n is a simple polytope of dimension $n - 1$. The number of d -dimensional faces is

$$f_d(\Pi_n) = (n - d)! S(n, r), \quad d = 0, 1, \dots, n - 1, \quad (3.7)$$

where $S(n, r)$ is the Stirling number of the second kind.

In particular, Π_n has $n!$ vertices, $(n - 1)n!/2$ edges, and $2^n - 2$ facets.

Proof. Simplicity follows from Theorem 3.7 applied to $[G] = (1^n)$, giving degree $R = n - 1$. By Theorem 3.8, d -dimensional faces of Π_n correspond to ordered partitions of J_n into $n - d$ blocks.

The number of such partitions equals $(n - d)! S(n, n - d)$, which is the classical formula for ordered Stirling numbers.

For $d = 0$: $f_0 = n! S(n, n) = n!$, confirming $|E_n| = n!$.

For $d = n - 2$ (facets): $f_{n-2} = 2! S(n, 2) = 2^n - 2$. ■

The normal fan of Π_n has an equally transparent combinatorial description. The normal cone of a face F corresponding to an ordered partition $(B_1, \dots, B_r)$ consists of all vectors $c \in \mathbb{R}^n$ such that $c_i = c_j$ whenever i, j belong to the same block B_r , and $c_i > c_j$ whenever $i \in B_r$ and $j \in B_s$ with $r < s$.

The normal fan of Π_n is therefore the *braid arrangement* of hyperplanes $\{x_i = x_j\}$ in $\mathbb{R}^n$: its $n!$ full-dimensional chambers correspond to total orders on J_n , and its codimension- r cones correspond to ordered partitions into r blocks.

Remark 3.4. The braid arrangement structure makes Π_n a *generalized permutohedron*: every deformation of Π_n that preserves the normal fan structure yields a polytope whose vertices are indexed by permutations and whose edges remain parallel to $e_i - e_j$. The multipermutohedra Π_{nk} of §3.2 are precisely such deformations, obtained by rescaling the right-hand sides of (3.6) according to the multiplicity structure of G .

The ordered-partition correspondence becomes concrete in small dimensions.

For $n = 3$, the permutohedron Π_3 is a regular hexagon in the plane $x_1 + x_2 + x_3 = 6$ in $\mathbb{R}^3$. It has 6 vertices, 6 edges, and 6 facets (in its 2-dimensional affine hull these are the 6 edges of the hexagon). The 6 ordered partitions of $J_3 = \{1, 2, 3\}$ into 3 singleton blocks correspond to the 6 vertices; the 6 ordered partitions into 2 blocks correspond to the 6 edges.

For $n = 4$, the permutohedron Π_4 is a truncated octahedron in the hyperplane $x_1 + x_2 + x_3 + x_4 = 10$ in $\mathbb{R}^4$. It has 24 vertices, 36 edges, and 14 facets (6 squares and 8 hexagons), and is a simple 3-dimensional polytope. Formula (3.7) confirms:

$$f_0 = 4! = 24, f_1 = 3 \cdot 24/2 = 36, f_2 = 2! \cdot S(4, 2) = 2 \cdot 7 = 14.$$

When coordinate repetitions are present in G , that is, when $[G] \neq (1^n)$, the ordered-partition description extends to multipermutation configurations and generalized permutohedra. Several ordered partitions may correspond to the same geometric face of Π_{nk} : if $n_i > 1$ for some i , then transpositions of two coordinates carrying the same value g_i do not change the point and do not correspond to edges of Π_{nk} .

The number of edges decreases from $(n - 1)n!/2$ to $R|E_{nk}|/2$ with R given by formula (3.8), and the face lattice of Π_{nk} is a quotient of the face lattice of Π_n by the equivalence induced by the primary specification $[G]$.

The results of this section establish a complete combinatorial-geometric model of the permutohedron. The ordered-partition correspondence, the f -vector

formula, the normal fan structure, and the hyperplane stratification together form a self-contained description of Π_n that serves as the canonical reference for all subsequent polyhedral constructions. In the following sections the analysis is extended to partial configurations, where partial occupancy and bounded repetitions introduce additional combinatorial structure.

§3.4. Partial Multipermutation Point Configurations

The configurations studied in §§3.2–3.3 are characterized by full occupancy: every admissible sequence of length n uses all elements of the coordinate realization G with their prescribed multiplicities. The present section introduces partial multipermutation point configurations, which generalize the full-occupancy regime by allowing sequences to select n elements from a multiset G of total multiplicity $\eta > n$, subject to bounded repetition constraints. This generalization produces a rich and stratified geometric structure that interpolates between the permutohedral regime of §3.2 and the product configurations of §3.7.

Throughout this section the coordinate realization G has underlying set $S(G) = \{g_1 < \dots < g_k\}$, primary specification $[G] = (n_1, \dots, n_k)$, and total multiplicity $\eta = n_1 + \dots + n_k > n$.

The condition $\eta > n$ distinguishes the partial regime from the full-occupancy case of §3.2. Each admissible sequence selects n elements from the ordered representation $A = (a_1 \leq \dots \leq a_\eta)$ subject to the constraint that element g_i appears at most n_i times. The case $n_i = 1$ for all i , so that $\eta = k$, corresponds to partial permutations without repetitions and is studied separately in §3.6.

The set of all admissible sequences is denoted $\tilde{P}_n^k$. Applying the immersion φ to $\tilde{P}_n^k$ yields the entire partial multipermutation point configuration

$$\tilde{E}_n^k(G) = \varphi(\tilde{P}_n^k) \subset \mathbb{R}^n.$$

The PMPC $\tilde{E}_n^k(G)$ reduces to the MPPC $E_{nk}(G)$ when $\eta = n$, and to the PPPC $E_n^k(G)$ when $\eta = k$. The case $n_i \geq n$ for all i leads to the UPMPC B_n^k treated in §3.7. Thus $\tilde{E}_n^k(G)$ interpolates between these extremal classes according to the structure of $[G]$.

Geometrically, a PMPC may be thought of as a union of MPPC strata, each lying in its own affine hyperplane and on its own hypersphere, assembled together into a higher-dimensional point set that no longer sits in a single hyperplane or on a single sphere. The organizing principle of this assembly is the *repetition vector*: a combinatorial label that records how many times each coordinate value appears in a given point, and thereby determines which stratum that point belongs to.

Formally, each point $x \in \tilde{E}_n^k(G)$ is associated with its repetition vector $\alpha(x) = (\alpha_1, \dots, \alpha_k)$, where α_i is the number of coordinates of x equal to g_i . Every admissible repetition vector satisfies

$$\sum_{i=1}^k \alpha_i = n, 0 \leq \alpha_i \leq n_i.$$

The set of all such vectors is denoted $A(G, n)$. For a fixed $\alpha \in A(G, n)$, the set of all points $x \in \tilde{E}_n^k(G)$ with $\alpha(x) = \alpha$ is called the α -stratum and is denoted V_α . It consists of all permutations of the vector containing α_i copies of g_i for each i , and is therefore an MPPC generated by the sub-multiset $G_\alpha = (g_1^{\alpha_1}, \dots, g_k^{\alpha_k})$.

Theorem 3.10. (*Stratum decomposition of PMPC*).

The entire PMPC decomposes into a disjoint union of strata:

$$\tilde{E}_n^k(G) = \bigcup_{\alpha \in A(G, n)} V_\alpha.$$

Each stratum V_α is an MPPC of the form $E_{n_\beta}^k(G_\alpha)$, where $\beta = \{i: \alpha_i > 0\}$ is the number of coordinate values actually used. In particular, each stratum is vertex-located and spherically located.

Proof. The repetition vector $\alpha(x)$ is well-defined for each $x \in \tilde{E}_n^k(G)$ and takes values in $A(G, n)$. The strata are disjoint by definition of $\alpha(x)$ and their union covers $\tilde{E}_n^k(G)$. Every point with repetition vector α is a permutation of the vector containing α_i copies of g_i , so $V_\alpha = E_{n_\beta}^k(G_\alpha)$.

By Theorem 3.4 applied to this sub-MPPC, each stratum is vertex-located. By Theorem 3.2, each stratum is spherically located. ■

Depending on α , the strata corresponding to different repetition vectors lie in different affine hyperplanes

$$\sum_{i=1}^k x_i = \sum_{i=1}^k \alpha_i g_i$$

and on different hyperspheres

$$\sum_{i=1}^k x_i^2 = \sum_{i=1}^k \alpha_i g_i^2.$$

Consequently the PMPC as a whole is not contained in any single affine hyperplane or hypersphere, in contrast to the full-occupancy MPPC.

Having identified the internal structure of $\tilde{E}_n^k(G)$, we turn to its cardinality. Since the strata are disjoint and each is an MPPC with a known cardinality formula, the total count follows by summing over all admissible repetition vectors.

For each $\alpha \in A(G, n)$ the stratum V_α has cardinality

$$\frac{n!}{\alpha_1! \cdots \alpha_k!}.$$

Summing over all admissible repetition vectors gives

$$|\tilde{E}_n^k(G)| = \sum_{\alpha \in A(G,n)} \frac{n!}{\alpha_1! \cdots \alpha_k!}.$$

The minimum cardinality is attained when all $n_i = 1$, that is, in the PPPC regime ($\eta = k$), giving

$$|\tilde{E}_n^k(G)| = \frac{k!}{(k-n)!}.$$

The maximum is attained when $n_i \geq n$ for all i , giving

$$|\tilde{E}_n^k(G)| = k^n,$$

corresponding to the UPMPC B_n^k .

The cardinality therefore satisfies

$$\frac{k!}{(k-n)!} \leq \tilde{E}_n^k(G) \leq k^n,$$

and varies monotonically with the repetition bounds encoded in $[G]$. Both bounds coincide with $n!$ in the full-occupancy PPC regime $k = n$.

Theorem 3.11. (*Hyperplane decomposition of PMPC*). For any subset $\omega \subset J_n$ with $|\omega| = s$, the configuration $\tilde{E}_n^k(G)$ decomposes as

$$\tilde{E}_n^k(G) = \bigcup_{t \in A(G,n)} \tilde{E}_n^k(G) \cap H_{t\omega},$$

where t ranges over all values attainable as partial sums

$$\sum_{i \in \omega} x_i$$

for $x \in \tilde{E}_n^k(G)$.

The minimum and maximum admissible values are

$$L_\omega = \sum_{j=1}^s g_j,$$

$$U_\omega = \sum_{j=k-s+1}^k g_j,$$

where $g_1 < \cdots < g_k$ is the ordered sequence of elements of $S(G)$ subject to multiplicity bounds.

Proof. Each point $x \in \tilde{E}_n^k(G)$ has a unique partial sum

$$t = \sum_{i \in \omega} x_i,$$

so the decomposition into layers is disjoint and covers $\tilde{E}_n^k(G)$.

The minimum L_ω is achieved by assigning to positions ω the s smallest available values subject to multiplicity bounds; the maximum U_ω by assigning the s largest. Both extremes are attained by admissible repetition vectors. ■

The extreme layers

$$\tilde{E}_n^k(G) \cap H_{L\omega, \omega}$$

and

$$\tilde{E}_n^k(G) \cap H_{U\omega, \omega}$$

lie on supporting hyperplanes of $\text{conv } \tilde{E}_n^k(G)$ and correspond to faces of the convex hull. The overall decomposition combines two complementary mechanisms: the combinatorial stratification by repetition vectors (Theorem 3.10) and the geometric stratification by hyperplanes. Their interaction governs the full polyhedral structure of the configuration and is the primary tool in the analysis of §3.5.

The stratification and hyperplane decompositions prepare the ground for the vertex analysis of the convex hull $\text{conv } \tilde{E}_n^k(G)$. In the full-occupancy case every configuration point is a vertex of its polytope. For PMPCs this is no longer true in general: points from different strata may be expressible as convex combinations of one another. The following necessary condition identifies which repetition vectors can possibly correspond to vertices, providing a combinatorial filter that will be sharpened into a complete criterion in §3.5.

Theorem 3.12. (*Necessary condition for vertexhood*).

A point $x \in \tilde{E}_n^k(G)$ with repetition vector $\alpha = \alpha(x)$ is a vertex of $\text{conv } \tilde{E}_n^k(G)$ only if α is extremal in $A(G, n)$: there is no pair of distinct vectors $\beta, \gamma \in A(G, n)$ such that $\alpha = (\beta + \gamma)/2$.

Proof. If $\alpha = (\beta + \gamma)/2$ for some $\beta \neq \gamma$ in $A(G, n)$, choose any $y \in V_\beta$ and $z \in V_\gamma$. The midpoint $(y + z)/2$ has coordinate i -marginal $(\beta_i + \gamma_i)/2 = \alpha_i$ for each i . By a symmetrization argument, one can choose y and z so that $x = (y + z)/2$, showing that x is not a vertex of $\text{conv } \tilde{E}_n^k(G)$. ■

The converse does not hold in general: extremality of α is necessary but not sufficient for vertexhood. A complete characterization of vertices of $\text{conv } \tilde{E}_n^k(G)$ is given in §3.5.

Remark 3.5. PMPCs inherit a natural symmetry structure from their coordinate realizations. Any permutation of coordinate indices is an isometry of $\mathbb{R}^n$ mapping $\tilde{E}_n^k(G)$ onto itself, so the symmetric group S_n acts on $\tilde{E}_n^k(G)$ by coordinate permutations. A PMPC is centrally symmetric when the realization G is symmetric about its weighted mean

$$\bar{a} = \frac{1}{\eta} \sum_{i=1}^k n_i g_i,$$

meaning that for every $g \in S(G)$ the reflected value $2\bar{a} - g$ also lies in $S(G)$ with the same multiplicity bound.

The combinatorial and stratification analysis developed in this section provides the foundation for the polyhedral study of PMPCs. The stratum decomposition (Theorem 3.10) identifies the internal building blocks of $\tilde{E}_n^k(G)$, the cardinality formula quantifies their collective size, the hyperplane decomposition (Theorem 3.11) reveals the layered geometric structure, and the necessary vertex condition (Theorem 3.12) prepares the ground for the complete vertex characterization. The convex hulls of PMPCs, their vertex structure, adjacency relations, and H-representations are analyzed in §3.5.

§3.5. Partial Multipermutohedra

The stratification and hyperplane decompositions established in §3.4 provide the combinatorial foundation for studying the convex hulls of partial multipermutation configurations. The present section introduces and analyzes the class of partial multipermutohedra, which arise as convex hulls of PMPCs and form a rich family of polytopes interpolating between classical permutohedra and hyperrectangles. The central results concern the vertex criterion, the adjacency structure, and the H-representation of these polytopes.

Throughout this section the coordinate realization G has underlying set $S(G) = \{g_1 < \dots < g_k\}$, primary specification $[G] = (n_1, \dots, n_k)$, and total multiplicity $\eta > n$. The convex hull of the entire PMPC is denoted

$$\tilde{\Pi}_n^k(G) = \text{conv } \tilde{E}_n^k(G)$$

and is called the *partial multipermutohedron* associated with G and n .

By construction $\tilde{\Pi}_n^k(G) \subset \mathbb{R}^n$ is a convex polytope whose combinatorial and facial structure depends on n , k , and the full primary specification $[G]$, making the family of partial multipermutohedra considerably richer than that of classical permutohedra.

Theorem 3.13. (*Dimension of the partial multipermutohedron*).

The partial multipermutohedron $\tilde{\Pi}_n^k(G)$ has dimension n if and only if the set of coordinate sums

$$\left\{ \sum_{i=1}^k \alpha_i g_i : \alpha \in A(G, n) \right\}$$

contains at least two distinct values. This holds whenever $k \geq 2$ and the multiplicity bounds do not force a unique repetition vector. When all coordinate

sums coincide, $\tilde{\Pi}_n^k(G)$ has dimension at most $n - 1$ and reduces to a multipermutohedron.

Proof. If the coordinate sums take at least two distinct values then the affine hull of $\tilde{E}_n^k(G)$ is not contained in any hyperplane

$$\sum_{i=1}^n x_i = c, \quad (3.8)$$

so the affine dimension of $\tilde{\Pi}_n^k(G)$ is not restricted by a coordinate-sum constraint.

The coordinate permutation symmetry then forces the affine hull to be all of $\mathbb{R}^n$, giving

$$\dim \tilde{\Pi}_n^k(G) = n.$$

If all coordinate sums coincide, all points of $\tilde{E}_n^k(G)$ lie in the hyperplane (3.8)

$$\dim \tilde{\Pi}_n^k(G) \leq n - 1$$

and the polytope reduces to the multipermutohedron of the unique stratum. ■

Degenerate cases arise when the repetition structure forces a unique distribution, for instance when G consists of a single value with sufficient multiplicity. In such situations $\tilde{\Pi}_n^k(G)$ may reduce to a single point. In the generic case $k \geq 2$ with at least two distinct coordinate sums, $\tilde{\Pi}_n^k(G)$ is a full-dimensional polytope in $\mathbb{R}^n$.

The vertex structure of $\tilde{\Pi}_n^k(G)$ is more subtle than in the full-occupancy case, because not every point of $\tilde{E}_n^k(G)$ is necessarily a vertex of the convex hull. Theorem 3.12 of §3.4 established a necessary condition: the repetition vector of a vertex must be extremal in $A(G, n)$. The following theorem provides a complete characterization.

Theorem 3.14. (*Vertex criterion for PMPC*).

A point $x \in \tilde{E}_n^k(G)$ with repetition vector $\alpha = \alpha(x)$ is a vertex of $\tilde{\Pi}_n^k(G)$ if and only if there exists a vector $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ such that the linear function

$$\langle c, x \rangle = \sum_{i=1}^n c_i x_i$$

is strictly maximized over $\tilde{E}_n^k(G)$ at x and at no other point.

Proof. The condition is equivalent to x being an extreme point of $\text{conv } \tilde{E}_n^k(G)$. By definition, x is a vertex of $\tilde{\Pi}_n^k(G)$ if and only if it cannot be written as a convex combination of other points of $\tilde{\Pi}_n^k(G)$.

Since $\tilde{E}_n^k(G)$ is finite, this is equivalent to the existence of a strictly separating linear functional. If $\alpha(x)$ is extremal in $A(G, n)$, choose generic strictly ordered coefficients $c_1 > \dots > c_n$. Then $\langle c, x \rangle$ is uniquely maximized over $\tilde{E}_n^k(G)$ by the lexicographically largest arrangement of the coordinates of x , which is x itself when the ordering is chosen appropriately. Conversely, if $\alpha(x)$ is not extremal, Theorem 3.12 shows x lies in the relative interior of a segment connecting two configuration points, so it is not a vertex. ■

In particular, every point $x \in \tilde{E}_n^k(G)$ whose repetition vector $\alpha(x)$ is a vertex of the polytope $\text{conv } A(G, n)$ (regarded as a subset of $\mathbb{R}^k$) is a vertex of $\tilde{\Pi}_n^k(G)$. For PPPC strata ($\eta = k$, all multiplicities one) every point is a vertex by this criterion, consistent with the fact that PPPCs are vertex-located within their own strata.

The criterion of Theorem 3.14 admits an equivalent geometric formulation in terms of supporting hyperplanes. A point $x \in \tilde{E}_n^k(G)$ is a vertex of $\tilde{\Pi}_n^k(G)$ if and only if there exists an affine hyperplane

$$H = \{y \in \mathbb{R}^n: \langle c, y \rangle = \langle c, x \rangle\}$$

such that $\langle c, y \rangle < \langle c, x \rangle$ for every $y \in \tilde{E}_n^k(G)$ with $y \neq x$.

This hyperplane supports $\tilde{\Pi}_n^k(G)$ at x , separating it from all other configuration points. Geometrically, x is a vertex precisely when it occupies an exposed extreme position within the polytope: the supporting hyperplane touches $\tilde{\Pi}_n^k(G)$ at x alone. This characterization makes clear why vertexhood depends on the repetition vector rather than on the coordinate ordering: two points with the same repetition vector α lie on the same sphere and in the same affine hyperplane, so no single linear functional can separate one from the other; only points whose repetition vectors are genuinely extremal in $A(G, n)$ can be isolated by supporting hyperplanes.

Having characterized the vertices, we turn to the edge structure of $\tilde{\Pi}_n^k(G)$. Two distinct vertices $x, y \in \tilde{E}_n^k(G)$ are *adjacent* in $\tilde{\Pi}_n^k(G)$ if the segment $[x, y]$ is an edge of the polytope. Adjacency in partial multipermutohedra admits two qualitatively different types depending on whether x and y belong to the same stratum or to different strata.

Theorem 3.15. (*Adjacency criterion*).

Two distinct vertices $x, y \in \tilde{E}_n^k(G)$ are adjacent in $\tilde{\Pi}_n^k(G)$ if and only if one of the following holds:

(i) *Intra-stratum adjacency.* x and y belong to the same stratum V_α and y is obtained from x by transposing two coordinates $x_i = g_p$ and $x_j = g_q$ with $g_p \neq g_q$, subject to the condition that no other configuration point lies on the segment $[x, y]$.

(ii) *Inter-stratum adjacency.* $x \in V_\alpha$ and $y \in V_\beta$ with $\alpha \neq \beta$, and the repetition vectors α and β differ by a single unit transfer: $\beta_p = \alpha_p - 1$ and $\beta_q = \alpha_q + 1$ for some $p \neq q$, with all other components equal, and $0 \leq \beta_i \leq n_i$ for all i . Furthermore, y is obtained from x by replacing one coordinate equal to g_p by g_q .

Proof. By the standard edge criterion, $[x, y]$ is an edge of $\tilde{\Pi}_n^k(G)$ if and only if there exists a linear functional strictly maximized over $\tilde{\Pi}_n^k(G)$ on $[x, y]$ and at no other point. In the intra-stratum case the analysis reduces to the stratum, and Theorem 3.7 of §3.2 applies directly.

In the inter-stratum case the repetition vectors differ; a single unit transfer is the minimal admissible change by the integrality of $A(G, n)$. Any larger change would imply an intermediate admissible repetition vector $\gamma = (\alpha + \beta)/2 \in A(G, n)$, which would place a configuration point in the relative interior of $[x, y]$ and contradict the edge property. ■

Intra-stratum edges are parallel to vectors of the form $e_i - e_j$ (inherited from the permutohedron structure of the stratum), while inter-stratum edges are parallel to vectors of the form $e_i(g_q - g_p)$ at the position where the coordinate replacement occurs. The degree of a vertex x with repetition vector α equals the number of admissible unit transfers, that is, the number of ordered pairs (p, q) with $\alpha_p > 0$ and $\alpha_q < n_q$. This number varies across vertices, so the adjacency graph of $\tilde{\Pi}_n^k(G)$ is in general non-regular.

This non-regularity distinguishes partial multipermutohedra from classical permutohedra, where every vertex has the same degree $n - 1$. The variation in degree reflects the combinatorial richness of the partial setting: vertices near the boundary of the admissible repetition region $A(G, n)$ have fewer neighbours, since fewer unit transfers are available without violating the multiplicity bounds, while vertices in the interior of $A(G, n)$ have the maximum number of admissible transfers and thus the highest degree. The adjacency structure induced by these two types of edges provides the combinatorial skeleton from which the facial structure of $\tilde{\Pi}_n^k(G)$ is built. In particular, the H-representation derived below can be understood as an analytic encoding of the same information: each inequality in the system corresponds to a family of parallel supporting hyperplanes whose active sets define faces of the polytope.

A complete analytic description of $\tilde{\Pi}_n^k(G)$ is provided by its H-representation. The bounds L_ω and U_ω introduced in Theorem 3.11 of §3.4 determine the facial inequalities of the polytope.

Theorem 3.16. (*H-representation of partial multipermutohedron*).

A point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ belongs to $\tilde{\Pi}_n^k(G)$ if and only if it satisfies the system of inequalities

$$L_\omega \leq \sum_{i \in \omega} x_i \leq U_\omega, \quad \omega \subset J_n, \quad 1 \leq |\omega| \leq n, \quad (3.9)$$

where

$$L_\omega = \sum_{j=1}^{|\omega|} g_j, \\ U_\omega = \sum_{j=k-|\omega|+1}^k g_j,$$

are the minimum and maximum partial sums from Theorem 3.11, and $g_1 < \dots < g_k$ is the ordered sequence of elements of $S(G)$ with appropriate multiplicity bounds. The inequalities for $|\omega| = n$ reduce to the bounds on the total coordinate sum.

Proof. Every point $x \in \tilde{E}_n^k(G)$ satisfies (3.9) by definition of L_ω and U_ω as extremes over the configuration. Since $\tilde{\Pi}_n^k(G) = \text{conv } \tilde{E}_n^k(G)$ and the constraints are linear, they hold for all convex combinations, so $\tilde{\Pi}_n^k(G)$ is contained in the

polytope P defined by (3.9). Conversely, any extreme point of P satisfies a maximal active subset of (3.9) and can be identified with a permutation of an admissible repetition vector, hence belongs to $\tilde{E}_n^k(G)$. Therefore $P = \tilde{I}_n^k(G)$. ■

Not all inequalities in (3.9) are facet-defining; some may be redundant depending on n , k , and $[G]$. In extremal regimes the representation simplifies: for PPPCs ($\eta = k$) the bounds reduce to those of the classical partial permutohedron; for UP-MPCs the inequalities decouple into coordinate-wise bounds $g_1 \leq x_i \leq g_k$, yielding the hyperrectangle representation of §3.7.

The H-representation (3.9) reveals the geometric essence of the partial multi-permutohedron. Each inequality

$$L_\omega \leq \sum_{i \in \omega} x_i \leq U_\omega$$

encodes a constraint on how coordinate values may be distributed among the positions indexed by ω : the partial sum over any subset of positions is bounded above by the maximum achievable with the s largest admissible values, and below by the minimum achievable with the s smallest. This is the analytic counterpart of the combinatorial stratification of §3.4: the hyperplane layers H_{t_ω} of Theorem 3.11 are precisely the level sets of the partial sum function

$$\sum_{i \in \omega} x_i,$$

and the extreme layers correspond to the active inequalities in (3.8) that define the facets of $\tilde{I}_n^k(G)$. In this sense the H-representation is not merely a formal description but a direct reflection of the two-level decomposition structure – combinatorial by repetition vectors, geometric by hyperplane layers – that was established as the organizing principle of the entire section.

Remark 3.6. The partial multipermutohedron $\tilde{I}_n^k(G)$ is centrally symmetric when the realization G is symmetric about its weighted mean

$$\bar{a} = \frac{1}{\eta} \sum_{i=1}^k n_i g_i :$$

for every $g \in S(G)$ the reflected value $2\bar{a} - g$ lies in $S(G)$ with the same multiplicity bound n_i .

Under this condition the mapping $x \mapsto 2\bar{a} \cdot 1 - x$, where 1 denotes the all-ones vector in $\mathbb{R}^n$, is an isometry of $\mathbb{R}^n$ that maps $\tilde{I}_n^k(G)$ onto itself. Centrally symmetric partial multipermutohedra admit antipodal vertex pairs and a correspondingly regular edge structure, whereas the asymmetric case may exhibit strongly non-uniform adjacency graphs.

The polyhedral analysis of this section completes the geometric characterization of partial multipermutation point configurations. Starting from the stratum de-

composition of §3.4, we have established the dimension of $\tilde{\Pi}_n^k(G)$ (Theorem 3.13), a complete vertex criterion (Theorem 3.14), a two-type adjacency characterization (Theorem 3.15), and an explicit H-representation (Theorem 3.16). Together these results demonstrate that partial multipermutohedra form a rich and flexible class of polytopes generalizing classical permutohedra in multiple directions. In the following section we consider the injective specialization of this framework, corresponding to the absence of coordinate repetitions, which leads to partial permutation point configurations and partial permutohedra with a structurally simpler description.

§3.6. Partial Permutation Point Configurations

The present section studies partial permutation point configurations (PPPCs), which constitute the injective specialization of the partial multipermutation framework: every admissible sequence selects n elements from the coordinate realization G without repetitions. This constraint produces configurations with a structurally simpler polyhedral description than the general PMPC, yet richer than the classical PPC. PPPCs occupy a central position in the hierarchy between full permutation configurations ($k = n$) and general partial multipermutation configurations ($\eta > k$).

Throughout this section the coordinate realization G is a plain set of pairwise distinct real numbers:

$$G = S(G) = \{g_1 < \dots < g_k\}, \quad [G] = (1, \dots, 1), \quad \eta = k, \quad k > n.$$

The set of all admissible sequences consists of all injective mappings

$$\pi : J_n \rightarrow S(G)$$

and the entire PPPC is

$$E_n^k(G) = \varphi(P_n^k) \subset \mathbb{R}^n.$$

Its cardinality equals the number of arrangements of n elements from k without repetition:

$$|E_n^k(G)| = \frac{k!}{(k-n)!}.$$

This is the lower bound in the cardinality inequality of §3.4, confirming that PPPCs are the minimal-cardinality members of the PMPC family for given n and k .

Since all multiplicities equal one, the admissible repetition vectors are exactly the 0 – 1 vectors $\alpha \in \{0,1\}^k$ with

$$\sum_{i=1}^k \alpha_i = n.$$

Since all multiplicities equal one, each point $x \in E_n^k(G)$ uses exactly n distinct elements of $S(G)$. The subset $T = G(x) \subset S(G)$ of coordinate values of x has cardinality exactly n and determines which stratum the point belongs to. Denoting by V_T the set of all points $x \in E_n^k(G)$ whose coordinate values form exactly the subset T , that is

$$V_T = \{x \in E_n^k(G) : G(x) = T\},$$

every point of $E_n^k(G)$ belongs to exactly one such stratum, and the strata are pairwise disjoint. Since every n -element subset $T \subset S(G)$ is realized by some point of $E_n^k(G)$, the union over all C_k^n subsets T of size n covers the entire configuration:

$$E_n^k(G) = \bigcup_{T \subset S(G), |T|=n} V_T$$

Each stratum V_T is a PPC generated by the sub-realization T and is therefore vertex-located and spherically located by Theorems 3.4 and 3.2. The partial permutohedron is

$$\Pi_n^k(G) = \text{conv } E_n^k(G).$$

Unlike MPPCs, the PPPC is not contained in a single affine hyperplane: different strata have different coordinate sums

$$\sigma(T) = \sum_{g_i \in T} g_i.$$

Consequently $\Pi_n^k(G)$ is a full-dimensional polytope in $\mathbb{R}^n$ for $k > n$, in contrast to the classical permutohedron which has dimension $n - 1$.

Theorem 3.17. (*Vertex criterion for PPPC*).

A point $x \in E_n^k(G)$ is a vertex of $\Pi_n^k(G)$ if and only if the omitted set $S(G) \setminus G(x)$ forms a contiguous block in the ordered sequence $g_1 < \dots < g_k$. The number of vertices of $\Pi_n^k(G)$ equals $(n + 1)!$, independently of $k \geq n + 1$.

Proof. *Sufficiency.* Suppose

$$S(G) \setminus G(x) = \{g_p, \dots, g_{\ell+k-n-1}\}$$

is a contiguous block, so $G(x)$ consists of the remaining n values of $S(G)$.

Choose a vector $c \in \mathbb{R}^n$ with strictly decreasing components $c_1 > \dots > c_n$. The linear function

$$\langle c, y \rangle = \sum_{i=1}^n c_i y_i$$

over $E_n^k(G)$ is maximized by assigning the largest available coordinate value to the position with the largest weight c_1 , the next largest to c_2 , and so on.

Among all n -element subsets $T \subset S(G)$, the subset $T = G(x)$ yields the largest value of $\langle c, y \rangle$ because $G(x)$ contains the n values of $S(G)$ that are not in the omitted contiguous block. Since the components of c are strictly decreasing, this maximizer is unique, so x is a vertex of $\Pi_n^k(G)$.

Necessity. Suppose $S(G) \setminus G(x)$ is non-contiguous: there exist $g_p < g_m < g_r$ with $g_p, g_r \notin G(x)$ and $g_m \in G(x)$. Replacing g_m in $G(x)$ by g_p yields a point $y \in E_n^k(G)$, and replacing g_m by g_r yields $z \in E_n^k(G)$. By choosing appropriate permutations of coordinates one can arrange that $x = (y + z)/2$, so x is not a vertex of $\Pi_n^k(G)$.

To count the vertices: contiguous blocks of size $k - n$ in the ordered set $S(G)$ begin at positions $g_1, g_2, \dots, g_{n+1}$, giving exactly $n + 1$ blocks regardless of k . Each block corresponds to $n!$ orderings of the selected n values. Therefore

$$|\text{vert } \Pi_n^k(G)| = (n + 1)n! = (n + 1)! \quad \blacksquare$$

For $k > n + 1$ only $(n + 1)!$ of the $k!/(k - n)!$ configuration points are vertices of $\Pi_n^k(G)$; the remaining points lie in relative interiors of faces and are expressible as convex combinations of vertex points.

Theorem 3.18. (*Vertex-locatedness of PPPC*).

The configuration $E_n^k(G)$ is vertex-located, that is,

$$E_n^k(G) = \text{vert } \Pi_n^k(G),$$

if and only if $k = n + 1$.

Proof. By Theorem 3.17, the number of vertices of $\Pi_n^k(G)$ equals $(n + 1)!$ for any $k \geq n + 1$, while

$$|E_n^k(G)| = \frac{k!}{(k - n)!}.$$

These two quantities coincide if and only if

$$\frac{k!}{(k - n)!} = (n + 1)!,$$

which holds if and only if $k = n + 1$. $\blacksquare$

Remark 3.6. The extremal case $k = n + 1$ admits an alternative characterization via coordinate projection. The configuration $E_n^{n+1}(G)$ consists of all n -tuples obtained by removing one element from the $(n + 1)$ element set $S(G)$ and permuting the remaining n elements. The map that appends the missing element as the $(n + 1)$ -th coordinate, using the affine relation

$$x_{n+1} = \sum_{i=1}^{n+1} a_i - \sum_{i=1}^n x_i$$

establishes a bijection between $E_n^{n+1}(G)$ and the full PPC $E_{n+1}(G)$.

Since $E_{n+1}(G)$ is vertex-located (Theorem 3.4) and the projection is affine, $E_n^{n+1}(G)$ is vertex-located as well. This projection approach also provides an explicit method for constructing convex extensions on $E_n^{n+1}(G)$ from extensions on $E_{n+1}(G)$, which is exploited in Chapter 4.

Theorem 3.19. (*Adjacency criterion for PPPC*).

Two distinct vertices $x, y \in E_n^k(G)$ are adjacent in $\Pi_n^k(G)$ if and only if one of the following holds:

(i) *Intra-stratum adjacency.* x and y belong to the same stratum V_T and y is obtained from x by transposing two coordinates $x_i = g_p$ and $x_j = g_q$, with no other configuration point on the segment $[x, y]$.

(ii) *Inter-stratum adjacency.* $x \in V_T$ and $y \in V_{T'}$ with $T \neq T'$, and T' is obtained from T by replacing one element $g_p \in T$ by one element $g_q \in S(G) \setminus T$, with y obtained from x by replacing one coordinate equal to g_p by g_q .

Proof. This is the specialization of Theorem 3.15 to $[G] = (1, \dots, 1)$. In case (i), Theorem 3.15 (i) applies directly. In case (ii), the repetition vectors $\alpha(x)$ and $\alpha(y)$ differ by a single unit transfer: $\beta_p = \alpha_p - 1 = 0$ and $\beta_q = \alpha_q + 1 = 1$. Since all multiplicities equal one, no intermediate vector $\gamma = (\alpha + \beta)/2$ lies in $A(G, n)$ (as it would require half-integer components), confirming the edge property. ■

The degree of a vertex $x \in V_T$ in $\Pi_n^k(G)$ equals $(n - 1) + n(k - n)$: the first term counts intra-stratum edges and the second counts inter-stratum edges (n positions each offering $k - n$ replacement values). The adjacency graph is non-regular in general, but in the extremal case $k = n + 1$ every vertex has degree $(n - 1) + n = 2n - 1$.

Theorem 3.20. (*H-representation of partial permutohedron*).

A point $x \in \mathbb{R}^n$ belongs to $\Pi_n^k(G)$ if and only if

$$\sum_{j=1}^s g_j \leq \sum_{i \in \omega} x_i \leq \sum_{j=k-s+1}^k g_j, \quad \omega \subset J_n, \quad s = |\omega|, \quad (3.10)$$

where $g^1 < \dots < g_k$ is the ordered sequence of elements of $S(G)$.

Proof. This is the specialization of Theorem 3.16 to $[G] = (1, \dots, 1)$. With all multiplicities one, the minimum partial sum

$$L_\omega = \sum_{j=1}^s g_j$$

and maximum

$$U_\omega = \sum_{j=k-s+1}^k g_j$$

are achieved by assigning the s smallest (resp. largest) values of $S(G)$ to positions ω . There is no affine equality constraint on the total sum since different strata have different coordinate sums, preserving full-dimensionality of $\Pi_n^k(G)$. ■

The facial structure of $\Pi_n^k(G)$ reflects the interaction of two combinatorial mechanisms: ordered partitions of coordinate indices (as in the classical permutahedron) and admissible selections of subsets of $S(G)$ (the occupancy structure). This dual layer makes the face lattice substantially more complex than that of Π_n , though many qualitative features are inherited. A complete description of the face lattice is deferred to Chapter 4.

The class of partial permutation point configurations is thus characterized by a clear separation between the vertex criterion (Theorem 3.17) and vertex-locatedness of the configuration as a whole (Theorem 3.18): individual vertices are identified by the contiguous-block condition, while the configuration is vertex-located only in the extremal case $k = n + 1$. This distinction is essential for the convex extension constructions of Chapter 4, where the vertex-located case $k = n + 1$ admits fully explicit analytic schemes via the projection technique of Remark 3.6. In the following section we consider the limiting case in which repetition constraints are removed entirely.

§3.7. Unbounded Configurations and Boolean Point Sets

The configuration classes studied in preceding sections were governed by explicit multiplicity bounds: each element of the coordinate realization G could appear at most n_i times in any admissible sequence. The present section removes all such bounds, allowing each coordinate position to take any value from $S(G)$ independently of the others. This limiting operation yields the simplest and most regular configurations in the hierarchy, yet they play a foundational role as reference models, bounding domains, and limiting cases that clarify the scope of the results obtained earlier.

Let G be a coordinate realization with $S(G) = \{g_1 < \cdots < g_k\}$ and $k \geq 2$. The unbounded regime is obtained by setting $n_i \geq n$ for every i , so that the repetition constraints become inactive. The set of admissible sequences then consists of all mappings

$$\pi: J_n \rightarrow S(G)$$

without any injectivity or multiplicity restriction. As defined in §3.1 (Definition 3.8), the resulting point configuration is called an *unbounded partial multi-permutation point configuration* (UPMPC) and is denoted

$$B_n^k(G) = S(G)^n = \varphi(P_n^\infty) \subset \mathbb{R}^n.$$

Each of the n coordinates independently ranges over the k values of $S(G)$, giving cardinality k^n . In the PMPC framework of §3.4, the UPMPC arises as the case where all multiplicity bounds are at least n , which causes the cardinality to reach its maximum value k^n .

Theorem 3.21. (*Structure of UPMPC*).

The UPMPC B_n^k has cardinality $|B_n^k| = k^n$ and its convex hull is the hyperrectangle

$$\Pi_n^\infty(G) = \text{conv } B_n^k(G) = [g_1^1, g_k^1]^n. \quad (3.11)$$

The vertices of $\Pi_n^\infty(G)$ are precisely the 2^n points whose every coordinate equals either g_1 or g_k , that is,

$$\text{vert } \Pi_n^\infty(G) = \{g_1, g_k\}^n = B_n^2(G).$$

Proof. The cardinality k^n follows since each coordinate independently takes k values. Since every $x \in B_n^k(G)$ satisfies $g_1 \leq x_i \leq g_k$ for all i , we have

$$B_n^k(G) \subset [g_1, g_k]^n.$$

The corners $(g_1, \dots, g_1)$ and $(g_k, \dots, g_k)$ both belong to $B_n^k(G)$, so $\text{conv } B_n^k(G) \supseteq [g_1, g_k]^n$, establishing (3.11). The vertices of the hyperrectangle $[g_1, g_k]^n$ are precisely its 2^n corners. All such corners belong to $B_n^k(G)$ since $\{g_1, g_k\} \subset S(G)$.

Any point $x \in B_n^k(G)$ with some coordinate $x_i = g_j$ for $1 < j < k$ lies in the relative interior of an edge of $\Pi_n^\infty(G)$, hence is not a vertex. ■

Theorem 3.21 identifies the vertices of $\Pi_n^\infty(G)$: they are exactly the points of the Boolean sub-configuration $B_n^2(G)$. This is the *vertex criterion* for UPMPC. The question of *vertex-locatedness* of the configuration $B_n^k(G)$ itself is separate and has a precise answer.

Theorem 3.22. (*Vertex-locatedness of UPMPC*). The configuration $B_n^k(G)$ is vertex-located, that is,

$$B_n^k(G) = \text{vert } \Pi_n^\infty(G),$$

if and only if $k = 2$.

Proof. By Theorem 3.21,

$$\text{vert } \Pi_n^\infty(G) = \{g_1, g_k\}^n = B_n^2(G).$$

The condition

$$B_n^k(G) = \text{vert } \Pi_n^\infty(G)$$

therefore holds if and only if

$$B_n^k(G) = B_n^2(G),$$

that is, $S(G) = \{g_1, g_k\}$, which is equivalent to $k = 2$. For $k > 2$ the configuration $B_n^k(G)$ contains points with intermediate coordinate values g_j ($1 < j < k$) that lie in the relative interior of faces of $\Pi_n^\infty(G)$ and are therefore not vertices. ■

Theorem 3.23.

(*H-representation of hyperrectangle*). A point $x \in \mathbb{R}^n$ belongs to $\Pi_n^\infty(G)$ if and only if

$$g^1 \leq x_i \leq g_k \quad (3.12)$$

for every $i \in J_n$.

This system of $2n$ inequalities is irredundant: each defines a facet of $\Pi_n^\infty(G)$.

Proof. The hyperrectangle $[g_1, g_k]^n$ is the intersection of the $2n$ half-spaces in (3.12). Removing any inequality strictly enlarges the feasible set, so the system is irredundant. Each inequality is active on a face of dimension $n-1$, confirming that it is a facet. ■

A structural feature distinguishing UPMPCs from bounded configurations is that the shape of $\Pi_n^\infty(G)$ depends only on the extreme values g_1 and g_k , regardless of intermediate values. The intermediate values $g_2, \dots, g_{k-1}$ contribute additional non-vertex points to $B_n^k(G)$ lying in the interior of faces of $\Pi_n^\infty(G)$, but do not change the polytope itself. The hyperplane decompositions of §3.4 (Theorem 3.11) take their most regular form here: for any $\omega \subset J_n$ every admissible partial sum actually occurs, each layer is isomorphic to $B_{n-s}^k(G)$, and the layer structure is uniform in contrast to bounded configurations.

The most important special case is the Boolean configuration. By Definition 3.9 of §3.1, the UPMPC with two-element realization

$$S(G) = \{b_1, b_2\}, \quad b_1 < b_2,$$

is called a *Boolean point configuration*, denoted

$$B_n^2(G) = \{b_1, b_2\}^n.$$

By Theorem 3.22 it is the unique UPMPC subclass that is vertex-located. Its convex hull

$$\text{conv } B_n^2(G) = [b_1, b_2]^n$$

is the *Boolean hypercube*, affinely equivalent to the unit cube $[0,1]^n$.

Theorem 3.24. (*Structure of Boolean configuration*).

The Boolean configuration $B_n^2(G)$ is vertex-located and centrally symmetric about the center $c = ((b_1 + b_2)/2) \cdot 1$. Its convex hull is an n -dimensional hypercube with 2^n vertices, $n2^{n-1}$ edges, and $2n$ facets.

Proof. Vertex-locatedness follows from Theorem 3.22 with $k = 2$. The f -vector of the n -dimensional hypercube is standard. Central symmetry: the realization $\{b_1, b_2\}$ is symmetric about $(b_1 + b_2)/2$, so the map

$$x \mapsto (b_1 + b_2) \cdot 1 - x$$

sends each vertex of $B_n^2(G)$ to its antipodal vertex. ■

Two vertices $x, y \in B_n^2(G)$ are adjacent in $[b_1, b_2]^n$ if and only if they differ in exactly one coordinate, which under the identification $b_1 \Leftrightarrow 0, b_2 \Leftrightarrow 1$ is the

Hamming adjacency condition. The adjacency graph of $B_n^2(G)$ is therefore the n -dimensional hypercube graph with uniform vertex degree n .

Remark 3.7. The uniform degree n of the hypercube graph contrasts with the non-uniform degrees in bounded configurations (Theorem 3.15). The regularity of $B_n^2(G)$ follows directly from full coordinate independence: any one coordinate may be flipped independently of the others.

Boolean configurations admit a polyhedral-spherical representation. All 2^n vertices of $[b_1, b_2]^n$ lie on the sphere centered at c with radius $r = ((b_2 - b_1)/2)\sqrt{n}$:

$$B_n^2(G) = [b^1, b^2]^n \cap S_{r(c)}. \quad (3.13)$$

This is the polyhedral-spherical representation (Theorem 3.5) in the two-level case. For general $B_n^k(G)$ with $k > 2$ the configuration is not spherically located, but the vertex subset $B_n^2(G) \subset B_n^k(G)$ still satisfies (3.13).

Remark 3.8. The hyperrectangle $\Pi_n^\infty(G) = [g_1, g_k]^n$ serves as a natural bounding box for all bounded configurations in the hierarchy. Since $\Pi_{nk}(G) \subset \Pi_n^\infty(G)$ and $\tilde{\Pi}_{nk}(G) \subset \Pi_n^\infty(G)$, any convex function defined on $\Pi_{nk}(G)$ restricts to an admissible extension on the sub-polytopes. This bounding-box property will be exploited systematically in Chapter 4.

The following table consolidates the principal configuration classes, their polytopes, and key geometric properties.

Table 3.2. Configuration classes and their polyhedral realizations

Class	Configuration	Polytope	Parameters	Vertex-located	Geometry
PPC	$E_n(G)$	$\Pi_n(G)$	$\eta = n = k$	yes	Permutohedron
MPPC	$E_{nk}(G)$	$\Pi_{nk}(G)$	$\eta = n \geq k$	yes	Multipermutohedron
PPPC	$E_n^k(G)$	$\Pi_n^k(G)$	$n < \eta = k$	iff $k = n+1$	Partial permutohedron
PMPC	$\tilde{E}_n^k(G)$	$\tilde{\Pi}_n^k(G)$	$n < \eta, k \leq \eta$	stratum-wise	Partial multipermutohedron
UPMPC	$B_n^k(G)$	$\Pi_n^\infty(G)$	$\eta_i \geq n$	iff $k = 2$	Hyperrectangle
Boolean	$B_n^2(G)$	$[b_1, b_2]^n$	$k = 2$	yes	Hypercube

The unbounded configurations $B_n^k(G)$ and their Boolean specialization $B_n^2(G)$ complete the hierarchy of combinatorial point configurations. Their simple polyhedral geometry, explicit H-representation, and transparent vertex and adjacency structure establish them as the maximally regular members of the hierarchy and as natural reference models for the convex extension theory of Chapter 4.

The configuration classes introduced in this chapter populate the abstract framework of Chapter 2 with concrete geometric objects. Vertex-locatedness holds unconditionally for permutation, multipermutation, and Boolean configurations, stratum-wise for partial multipermutation configurations, and for partial

permutation configurations if and only if the sequence length equals the ground set cardinality plus one. The polyhedral-spherical representations identified for full and Boolean configurations provide the geometric carriers on which strongly convex and differentiable extensions are realized, while the projection technique available in the extremal partial permutation case supplies an explicit construction route for that boundary class.

The vertex-located and polyhedral-spherical properties established here make the full apparatus of convex extension theory directly applicable to the combinatorial optimization problems studied in Chapters 4 and 5. The explicit polyhedral descriptions of the carrier polytopes, together with the adjacency and vertex criteria developed for each class, supply the structural data required for the constructive extension schemes and extremal ordering principles of Chapter 4.

Chapter 4

Extremal Properties and Convex Extensions on Combinatorial Point Configurations

The geometric framework developed in Chapter 3 establishes that multipermutation, partial permutation, and unbounded configurations are vertex-located and admits explicit polyhedral descriptions of their convex hulls. This makes the convex extension theory of Chapter 2 directly applicable to functions defined on these configurations. The present chapter realizes this applicability concretely, constructing explicit extension schemes and deriving extremal principles governing discrete optimization on combinatorial domains.

The chapter opens with constructive convex extension schemes for functions on full multipermutation point configurations (§4.1). Polynomial, quadratic, and strongly convex extensions are built explicitly, transferring discrete optimization problems to convex geometric domains without destroying their extremal structure. Building on this foundation, extremal ordering rules are derived for linear, separable, and distance-type objectives in §4.2, establishing which combinatorial arrangements minimize or maximize such functions over multipermutation configurations. The analysis is then extended to general functions via convex continuation arguments in §4.3, where optimality conditions are formulated in terms of the gradient structure of the extension.

The second part of the chapter addresses partial configurations. For partial permutation point configurations, extremal properties of separable and quadratic functionals are established via exchange mechanisms and reductions to classical assignment problems (§4.4), and lower bounds for convex extensions are derived together with sufficient optimality conditions (§4.5). Unbounded partial multipermutation configurations are treated in §4.6, where explicit convex extension schemes are constructed by exploiting the hyperrectangular geometry of their convex hulls. The chapter concludes with special subclasses admitting particularly explicit analytic schemes via projection techniques (§4.7).

Together, the results of this chapter establish a systematic correspondence between combinatorial geometry and convex analysis. The extension schemes and extremal ordering principles developed here provide the analytic foundation for the optimization methods constructed in Chapter 5.

§4.1. Convex Extension Schemes on Multipermutation Point Configurations

We begin by transferring the convex extension framework developed in Chapter 2 to combinatorial point configurations generated by numerical realizations of ordered combinatorial objects.

We consider a finite numerical multiset

$$G = \{g_1^{n_1}, \dots, g_k^{n_k}\} \subset \mathbb{R},$$

inducing a multipermutation point configuration (MPPC) of length n . The values $g_1, \dots, g_k$ are assumed to be pairwise distinct and ordered in strictly increasing order, $g_1 < \dots < g_k$, while the multiplicities satisfy $n_1 + n_2 + \dots + n_k = n$.

The MPPC generated by G is denoted $E_{nk}(G) \subset \mathbb{R}^n$ and consists of all coordinate permutations of the multiset G .

For analytic constructions, it is convenient to pass to the expanded ordered representation of the multiset. Listing all elements of G with repetitions and arranging them in nondecreasing order, we obtain

$$a_1 \leq \dots \leq a_n,$$

where each value g_r appears exactly n_r times.

The ordered sequence $(a_1, \dots, a_n)$ represents the expanded realization of the multiset G in which each support value g_r appears according to its multiplicity n_r . This representation will be used throughout the analytic constructions that follow.

Since the point configuration $E_{nk}(G)$ is vertex-located (*Theorem 3.4*), its points coincide with the vertices of its convex hull, namely the multipermutohedron

$$\Pi_{nk}(G) = \text{conv } E_{nk}(G) \subset \mathbb{R}^n.$$

Consequently, the theory of convex extensions developed in Chapter 2 is directly applicable to functions defined on any convex superset X containing $E_{nk}(G)$, in particular on

$$X = \Pi_{nk}(G).$$

To avoid cluttering formulas, we will henceforth use the abbreviated notations

$$E_{nk} = E_{nk}(G), \quad \Pi_{nk} = \Pi_{nk}(G).$$

The theory established there, however, is existential in nature. It ensures the possibility of extending functions while preserving convexity properties, but it does not provide explicit analytic constructions of such extensions. For applications in extremal analysis and discrete optimization, constructive realizations are essential, since practical estimates, stability analysis, and optimization procedures require concrete extension formulas rather than abstract existence results.

For this reason, in the present section we focus on constructive schemes for convex extension on MPPCs. To obtain analytically tractable constructions, we restrict attention to polynomial functions defined on E_{nk} . Polynomial functions form an algebraically stable class that is closed under addition, multiplication, and symmetrization operations induced by permutation structures. These closure properties make polynomial models especially suitable for constructive continuation procedures: extension operators are developed first for elementary monomials and then composed to handle more complex functional forms. Moreover, the extremal theorems for general objective functions established in the following paragraphs rely on continuation structures first built for polynomial models.

Let $f(x)$ be a function defined on the MPPC E_{nk} under the assumption that $0 < a_1 \leq \dots \leq a_n$,

In what follows we restrict attention to the case where f admits a polynomial representation in the coordinate variables. For this, we use a form of $f(x)$ as a sum of monomials of degree m :

$$f(x) = \sum c_{l_1 \dots l_n} F_{l_1 \dots l_n}^{(m)}(x),$$

where

$$F_{l_1 \dots l_n}^{(m)}(x) = \prod x_i^{l_i}$$

and the summation is taken over all multi-indices $\{l_1, \dots, l_n\}$ such that $l_1 + \dots + l_n = m$ and $l_i \geq 0$ for all i .

It should be noted that symmetric polynomials take constant values at all points of E_{nk} . This property follows from the invariance of coordinate multisets under permutations. In particular, for any integer $m \geq 1$ one has the power-sum identity

$$\sum_{i=1}^n x_i^m = \sum_{i=1}^n a_i^m.$$

In particular, isolating any single term from this identity yields:

$$-x_k^m = \sum_{\substack{k=1 \\ k \neq i}}^n x_k^m - \sum_{k=1}^n a_k^m.$$

Hence all power sums remain constant across all points of MPPC E_{nk} .

We now show that monomials with negative coefficients can be represented, on the MPPC E_{nk} , as linear combinations of monomials with non-negative coefficients supplemented by additive constants arising from invariant identities.

Let $P(l_1, \dots, l_n)$ denote the set of all permutations of the exponent vector $(l_1, \dots, l_n)$, where $l_1 + \dots + l_n = m$ and $l_i \geq 0$ for all i .

Then for $x \in E_{nk}$, the symmetric identity holds

$$\sum_{(k_1, \dots, k_n) \in P(l_1, \dots, l_n)} \prod_{i=1}^n x_i^{k_i} = \lambda_{l_1, \dots, l_n}^{(m)},$$

where the constants are

$$\lambda_{l_1, \dots, l_n}^{(m)} = \sum_{(k_1, \dots, k_n) \in P(l_1, \dots, l_n)} \prod_{i=1}^n a_i^{k_i}.$$

Consequently, for points $x \in E_{nk}$, any monomial with negative sign may be rewritten as

$$-F_{k_1, \dots, k_n}^{(m)}(x) = \sum_{(l_1, \dots, l_n) \neq (k_1, \dots, k_n)} F_{l_1, \dots, l_n}^{(m)}(x) - \lambda_{k_1, \dots, k_n}^{(m)}.$$

These invariant relations imply that any symmetric polynomial combination reduces to a constant on the configuration. As consequence, monomials with negative coefficients may be represented through sums of monomials with nonnegative coefficients supplemented by additive constants determined by invariant identities.

Therefore, it suffices to construct convex extensions for monomials with non-negative coefficients.

We therefore proceed to an explicit constructive scheme for convex extensions of monomials defined on E_{nk} with $0 < a_1 \leq \dots \leq a_n$ onto the non-negative orthant of the Euclidean space $\mathbb{R}_{\geq 0}^n$.

For each monomial $F_{l_1, \dots, l_n}^{(m)}(x)$ we construct a convex function $\tilde{F}_{l_1, \dots, l_n}^{(m)}(x)$ such that

$$\tilde{F}_{l_1, \dots, l_n}^{(m)}(x) = F_{l_1, \dots, l_n}^{(m)}(x) \text{ for } x \in E_{nk}, \quad (4.1)$$

$$\tilde{F}_{l_1, \dots, l_n}^{(m)}(x) \geq 0 \text{ for } x \in \mathbb{R}_{\geq 0}^n. \quad (4.2)$$

The construction of convex extensions proceeds inductively with respect to the degree m , beginning with elementary monomials and extending stepwise to higher-degree products.

For $m = 1$, one obtains linear monomials of the form

$$\tilde{F}_{l_1, \dots, l_n}^{(1)}(x) = F_{l_1, \dots, l_n}^{(1)}(x) = x_j,$$

where $l_j = 1, l_i = 0 \forall i \in J_n / \{j\}$.

Linear terms are convex on $\mathbb{R}_{\geq 0}^n$ without modification.

Let $m = 2$. For quadratic monomials, consider products of the forms

$$F_{l_1, \dots, l_n}^{(2)}(x) = x_i^2, \text{ if } l_i = 2,$$

$$F_{l_1 \dots l_n}^{(2)}(x) = x_i x_j, \text{ if } l_i = l_j = 1, i \neq j.$$

We should note that the points $x \in E_{nk}$ satisfy the relations

$$\begin{aligned} x_i x_j &= \frac{1}{2}((x_i + x_j)^2 - x_i^2 - x_j^2) = \\ &= \frac{1}{2} \left((x_i + x_j)^2 + \sum_{\substack{s=1 \\ s \neq i, j}}^n x_s^2 - \lambda_{20 \dots 0}^{(2)} \right), \end{aligned} \quad (4.3)$$

where

$$\lambda_{20 \dots 0}^{(2)} = \sum_{s=1}^n a_s^2.$$

Let us denote the function on the right-hand side of (4.3) by $w_{ij}^{(2)}(x)$.

Since $x \in \mathbb{R}_{\geq 0}^n$, for $x \in E_{nk}$ we have

$$x_i x_j = \max\{0, w_{ij}^{(2)}(x)\}.$$

Let us consider the functions

$$\tilde{F}_{l_1 \dots l_n}^{(2)}(x) = x_i^2, \text{ if } l_i = 2,$$

$$\tilde{F}_{l_1 \dots l_n}^{(2)}(x) = \max\{0, w_{ij}^{(2)}(x)\}, \text{ if } l_i = l_j = 1, i \neq j.$$

One verifies that for $x \in E_{nk}$ holds

$$\tilde{F}_{l_1 \dots l_n}^{(2)}(x) = F_{l_1 \dots l_n}^{(2)}(x) \geq 0,$$

where $l_1 + \dots + l_n = 2$.

Assume convex extensions $\tilde{F}_{l_1 \dots l_n}^{(m)}(x)$ are constructed for m and satisfy (4.1)–(4.2) for all monomials $F_{l_1 \dots l_n}^{(m)}(x)$ of degree m for any $l_1 + \dots + l_n = m$.

We now construct a convex extension $\tilde{F}_{l_1 \dots l_n}^{(m+1)}(x)$ of degree $m+1$ for $l_1 + \dots + l_n = m+1$.

We will choose any j for which $l_j \geq 1$. Then

$$F_{l_1 \dots l_n}^{(m+1)}(x) = x_j F_{l_1 \dots (l_j-1) \dots l_n}^{(m)}(x).$$

Using

$$uv = \frac{1}{2}((u+v)^2 - u^2 - v^2)$$

and invariant identities, one obtains on E_{nk} the representation

$$\begin{aligned}
 F_{l_1 \dots l_n}^{(m+1)}(x) &= x_j F_{l_1 \dots (l_j-1) \dots l_n}^{(m)}(x) = \\
 &= \frac{1}{2} \left\{ (x_j + F_{l_1 \dots (l_j-1) \dots l_n}^{(m)}(x))^2 - x_j^2 - (F_{l_1 \dots (l_j-1) \dots l_n}^{(m)}(x))^2 \right\} = \\
 &= \frac{1}{2} \left\{ (x_j + \tilde{F}_{l_1 \dots (l_j-1) \dots l_n}^{(m)}(x))^2 + \sum_{\substack{i=1 \\ i \neq j}}^n x_i^2 - \lambda_{20 \dots 0}^{(2)} - \lambda_{2l_1 \dots 2l_j-2 \dots 2l_n}^{(2)} \right. \\
 &\quad \left. + \sum_{(k_1 \dots k_n) \in \mathcal{P}\{l_1 \dots l_j-1 \dots l_n\} / (l_1 \dots l_j-1 \dots l_n)} (\tilde{F}_{l_1 \dots l_j-1 \dots l_n}^{(m)}(x))^2 \right\} \quad (4.4)
 \end{aligned}$$

Denote the right-hand side of the expression (4.4) by $\omega^{(m+1)}(x)$. This function is convex on $\mathbb{R}_{\geq 0}^n$.

Since

$$\tilde{F}_{l_1 \dots (l_j-1) \dots l_n}^{(m)}(x) = F_{l_1 \dots (l_j-1) \dots l_n}^{(m)}(x)$$

on E_{nk} by the inductive hypothesis, one verifies directly that

$$\omega^{(m+1)}(x) = F_{l_1 \dots l_n}^{(m+1)}(x)$$

for all $x \in E_{nk}$, so the interpolation condition (4.1) is satisfied at step $m+1$.

Since the monomial $F_{l_1 \dots l_n}^{(m+1)}(x)$ is nonnegative on $\mathbb{R}_{\geq 0}^n$, then for $x \in E_{nk}$,

$$\tilde{F}_{l_1 \dots l_n}^{(m+1)}(x) = \max\{0, \omega^{(m+1)}(x)\} \geq 0.$$

Proceeding inductively, one constructs convex extensions for monomials of arbitrary degree and for polynomial functions on the MPPC.

The constructions above guarantee convex extensions of functions defined on E_{nk} . However, in many extremal problems convexity alone is insufficient to ensure uniqueness and stability of minimizers.

For this reason it is useful to strengthen the continuation so that the resulting extension becomes strongly convex.

The strongly convex extension is based on the following augmentation principle. Let $\tilde{F}(x)$ be any convex extension of a function prescribed on E_{nk} . Suppose there exists a strongly convex function $h(x)$ such that

$$h(x) = 0 \text{ for all } x \in E_{nk}.$$

Then, for any $\rho > 0$, the function

$$\Psi(x) = \tilde{F}(x) + \rho h(x)$$

is a strongly convex extension of the original function, because adding a strongly convex term preserves strong convexity, while the vanishing condition guarantees interpolation on E_{nk} .

For MPPCs, a natural choice of such a vanishing curvature term is induced by the quadratic power-sum invariant

$$\sum_{i=1}^n x_i^2 = \sum_{i=1}^n a_i^2, \quad x \in E_{nk}.$$

Define

$$h(x) = \sum_{i=1}^n x_i^2 - \sum_{i=1}^n a_i^2.$$

Then $h(x) = 0$ on E_{nk} .

The function

$$\sum_{i=1}^n x_i^2$$

is strongly convex on $\mathbb{R}^n$ with modulus 2 since its Hessian equals $2I$.

Subtracting the constant

$$\sum_{i=1}^n a_i^2$$

does not affect the Hessian, so $h(x)$ is strongly convex with the same modulus.

Therefore,

$$\Psi(x) = \tilde{F}(x) + \rho \left(\sum_{i=1}^n x_i^2 - \sum_{i=1}^n a_i^2 \right), \quad \rho > 0,$$

is a strongly convex extension on the carrier domain and coincides with the original function on E_{nk} . The parameter ρ controls the curvature intensity and can be used to strengthen extremal stability without changing the prescribed configuration values.

The constructions above provide convex and strongly convex extensions on nonnegative carrier domains, but they are generally nonsmooth due to truncation operators and curvature augmentation.

In applications where differentiability is required, an alternative construction becomes desirable. Such a construction is available under the strict positivity assumption $a_i > 0$, $i \in J_n$ and is based on logarithmic linearization of multiplicative invariants. We now turn to this smooth extension scheme.

Assume that all elements of the generating multiset are strictly positive so that all configuration points lie in the strictly positive orthant $\mathbb{R}_{>0}^n$. Under this

assumption, additional invariant relations arise that are unavailable in the merely nonnegative case.

To construct a differentiable convex extension for a monomial $F_{l_1 \dots l_n}^{(m)}(x)$ on E_{nk} we can offer the following approach.

Let us denote

$$\beta = \prod_{i=1}^n a_i.$$

Since for points $x \in E_{nk} \subset \mathbb{R}_{>0}^n$

$$\prod_{i=1}^n x_i = \prod_{i=1}^n a_i$$

then the equality is as follows

$$\sum_{i=1}^n \ln x_i = \ln \beta.$$

Hence, assuming

$$m = \sum_{i=1}^n l_i, \quad \mu = \max_{i \in J_n} l_i,$$

we have

$$\begin{aligned} \tilde{F}_{l_1 \dots l_n}^{(m)}(x) &= \exp \left(\sum_{i=1}^n l_i \ln x_i \right) = \\ &= \exp \left(\mu \ln \beta - \sum_{i=1}^n (\mu - l_i) \ln x_i \right) = \beta^\mu \prod_{i=1}^n x_i^{l_i - \mu}. \end{aligned}$$

This function is infinitely differentiable on $\mathbb{R}_{>0}^n$ and coincides with the original monomial on the MPPCE_{nk} .

Convexity follows from the fact that the logarithm of the extension is an affine function of logarithmic coordinates, while the exponential mapping preserves convexity under such transformations.

Thus, for strictly positive configurations, monomials admit smooth convex extensions constructed through logarithmic linearization of multiplicative invariants. By linearity, convex extensions of arbitrary polynomial functions are obtained by finite summation of the corresponding monomial extensions. The resulting extensions are smooth on $\mathbb{R}_{>0}^n$ and coincide with the original polynomial on the configuration carrier.

This logarithmic construction complements the inductive nonsmooth scheme developed earlier. While the inductive method is universally applicable, including degenerate configurations containing zero coordinates, the logarithmic scheme provides differentiable convex extensions under strict positivity assumptions, thereby enriching the analytic extension framework available for MPPCs. Together, the two constructions form a complete constructive framework for convex extension of polynomial functions on MPPCs across both nonnegative and strictly positive carrier domains.

In the quadratic case, the continuation can be written in an explicit closed form using symmetric invariants of the configuration, without relying on iterative truncation. This explicit representation is also important in applications, since many discrete optimization models involve quadratic objectives. For these reasons, we now examine quadratic functions separately as a transparent special case of the general continuation scheme.

Consider a quadratic function defined on the configuration by

$$f(x) = (Cx, x) + (b, x),$$

where $x = (x^1, \dots, x_n) \in \mathbb{R}^n$, $b = (b_1, \dots, b_n)$, and C is a symmetric $n \times n$ matrix.

To construct a convex extension of f to the whole space $\mathbb{R}^n$, we exploit quadratic identities that hold for all points $x \in E_{n_k}$. In particular,

$$-x_k^2 = \sum_{\substack{k=1 \\ k \neq i}}^n x_k^2 - \sum_{k=1}^n a_k^2.$$

and

$$\pm x_i x_j = \frac{1}{2} \left((x_i \pm x_j)^2 + \sum_{\substack{k=1 \\ k \neq i, j}}^n x_k^2 - \sum_{k=1}^n a_k^2 \right), 1 \leq i < j \leq n.$$

Using these identities and performing equivalent transformations of the quadratic form, we obtain the convex extension

$$\tilde{f}(x) = f(x) + \gamma \left(\sum_{i=1}^n x_i^2 - \sum_{i=1}^n a_i^2 \right),$$

where the coefficient $\tilde{b}\gamma$ is chosen as

$$\gamma = \sum_{i: c_{ii} < 0} |c_{ii}| + \sum_{i < j} |c_{ij}|.$$

Equivalently, the extension admits the canonical representation

$$\tilde{f}(x) = (\tilde{C}x, x) + (b, x) + d,$$

with

$$\tilde{C} = C + \gamma I, \quad d = -\gamma \sum_{i=1}^n a_i^2.$$

Since the correction term

$$\gamma \left(\sum_{i=1}^n x_i^2 - \sum_{i=1}^n a_i^2 \right)$$

vanishes on E_{nk} , one has $\tilde{f}(x) = f(x)$ for all $x \in E_{nk}$.

Moreover, $\tilde{C}$ is positive definite by construction, hence $\tilde{f}$ is strongly convex on $\mathbb{R}^n$.

The extension constructions developed above transform functions defined on discrete configurations into convex functions on continuous domains. This transformation is fundamental for extremal analysis, because convexity allows one to apply tools of continuous optimization to problems originally posed on discrete combinatorial structures.

In the next two paragraphs, we turn to the investigation of extremal properties of functions defined directly on MPPCs. Particular attention will be devoted to linear, separable, and quadratic objective functions, as well as to general convex extremality principles induced by permutation symmetries and vertex-located structure.

§4.2. Extremal Properties of Special Functions on Multipermutation Point Configurations

In this section we study extremal properties of functions defined on multipermutation point configurations (MPPC), continuing the convex-extension framework developed in §4.1. While convex extensions allow optimization problems to be transferred to convex geometric domains, many extremal properties can be analyzed directly on the configurations themselves.

Within the hierarchy of combinatorial point configurations introduced in Chapter 3, MPPC form the most general permutation-type class generated by numerical multisets with prescribed multiplicities. Permutation point configurations arise as the nondegenerate case corresponding to unit multiplicities, so the extremal principles established below apply to both settings and are formulated in terms of the expanded ordered sequence of the inducing multiset.

Let $G = \{a_1, \dots, a_n\}$ be the inducing numerical multiset generating the configuration. Listing its elements with repetitions and arranging them in nondecreasing order, we obtain the expanded ordered sequence

$$a_1 \leq \dots \leq a_n. \quad (4.5)$$

The multipermutation point configuration induced by G consists of all vectors $x = (x_1, \dots, x_n)$ obtained by permuting the ordered sequence $(a_1, \dots, a_n)$ among coordinate positions. In what follows, we use the abbreviated notation $E_{nk} = E_{nk}(G)$.

The central structural feature exploited throughout this section is that extremality criteria for vertices of Π_{nk} can be expressed through ordering relations between coordinates and objective coefficients, leading to explicit rearrangement-type solutions.

We begin with linear objective functions. Despite their apparent simplicity, linear models play a foundational role in the theory. Extremal properties of linear functions on multipermutation configurations generate explicit ordering rules of rearrangement type and form the analytic basis for subsequent estimates of minima of convex and strongly convex functions on multipermutohedra.

Consider a linear function

$$\ell(x) = \sum_{i=1}^n c_i x_i \quad (4.6)$$

where $c = (c_1, \dots, c_n) \in \mathbb{R}^n$, and the optimization problem

$$\min_{x \in E_{nk}} \ell(x).$$

Without loss of generality, assume that the elements of the inducing multiset are ordered as (4.5).

The solution is given by the following permutation rearrangement rule, which plays a central role in the extremal analysis of broader function classes considered below.

Lemma 4.1. (*Permutation rearrangement rule for linear functions*)

Let $\ell(x)$ be a linear function in the form (4.6) defined on the multipermutation point configuration E_{nk} . Then the minimum of $\ell(x)$ over E_{nk} is attained at a point $x^* = (x_1^*, \dots, x_n^*)$ corresponding to a permutation $(\pi_1, \dots, \pi_n)$ of $\{1, \dots, n\}$ such that

$$c_{\pi_1} \geq \dots \geq c_{\pi_n}, \quad (4.7)$$

$$x_{\pi_1}^* \leq \dots \leq x_{\pi_n}^*, \quad (4.8)$$

with

$$x_{\pi_i}^* = a_i, \quad i = 1, \dots, n.$$

Proof. Consider an arbitrary point $x \in E_{nk}$. Suppose there exist indices p and q such that $c_p > c_q$ and $x_p > x_q$.

Let $\tilde{x}$ be obtained from x by swapping the p -th and q -th coordinates, that is, $\tilde{x}_p = x_q$, $\tilde{x}_q = x_p$, while all other coordinates remain unchanged. Then

$$l(\tilde{x}) - l(x) = (c_p - c_q)(x_p - x_q).$$

Since $c_p - c_q > 0$ and $x_p - x_q > 0$, it follows that $l(\tilde{x}) < l(x)$.

Thus, whenever a larger coefficient corresponds to a larger coordinate value, swapping the coordinates strictly decreases the value of the linear function. Repeating this operation finitely many times yields a point for which coefficients ordered nonincreasingly correspond to coordinates ordered nondecreasingly. This process terminates when no inversion $c_p > c_q$ with $x_p > x_q$ remains, which is equivalent to conditions (4.7)–(4.8). Such a point realizes the minimum of $l(x)$ on E_{nk} . ■

Corollary (*Linear minimum on the multipermutohedron*)

The minimum of a linear function over the multipermutohedron $\Pi_{nk} = \text{conv } E_{nk}$ is attained at the same permutation point x^* that minimizes the function over the configuration E_{nk} .

Proof. Since Π_{nk} is the convex hull of the finite set E_{nk} and $l(x)$ is linear, its minimum over Π_{nk} is necessarily attained at a vertex of the polytope. ■

Beyond its geometric meaning, Lemma 4.1 admits an important interpretation in discrete optimization. The problem of minimizing a linear function over E_{nk} may be viewed as a special case of the assignment problem, where coefficients c_i play the role of positional weights and the elements a_i represent assignable values.

In this interpretation, the permutation rule corresponds to an optimal assignment obtained by matching the largest coefficients with the smallest available values. This establishes a direct connection between extremal properties of linear functions on multipermutation configurations and classical combinatorial optimization models.

The importance of this connection lies in the fact that similar assignment-type mechanisms arise in the analysis of broader function classes. In particular, separable convex functions and certain quadratic objectives may often be reduced to linear extremal problems through local linearization or decomposition arguments.

Remark 4.1. The permutation rule for linear functions is purely combinatorial and does not depend on convexity assumptions. Nevertheless, it plays a crucial role in convex analysis, since linear functions arise naturally as supporting functionals and first-order approximations of convex and strongly convex objectives.

The extremal principle established for linear objective functions extends naturally to a significantly broader and practically important class of optimization problems. In many applications, the objective admits a separable structure, in which the contribution of each coordinate is described by an individual one-dimensional function. Such models remain closely aligned with the permutation structure of the feasible set and naturally lead to assignment-type formulations

while preserving the possibility of explicit extremal characterization under suitable structural assumptions.

We therefore consider objective functions of the form

$$f(x) = \sum_{i=1}^n f_i(x_i), \quad (4.9)$$

where each function $f_i: \mathbb{R} \rightarrow \mathbb{R}$ is defined on the real line.

Optimization problems with separable objective functions arise in numerous applications and admit a natural interpretation in terms of assignment and allocation models. The extremal ordering principle established for linear objectives extends naturally to this broader setting: although coordinate contributions become nonlinear, their separable structure preserves the permutation symmetry of the feasible set while incorporating curvature effects into the extremal analysis.

Denote by $P(n)$ the set of all permutations of $\{1, \dots, n\}$.

Lemma 4.2. (*Separable minimization as an assignment problem*).

Let the function $f(x)$ given by (4.9) be defined on the multipermutation point configuration E_{nk} . Then

$$\min_{x \in E_{nk}} f(x) = \min_{\pi \in P(n)} \sum_{i=1}^n f_i(a_{\pi_i}),$$

where the minimum on the right is taken over all permutations $\pi = (\pi_1, \dots, \pi_n)$ of $\{1, \dots, n\}$.

Proof. Every point $x \in E_{nk}$ is obtained by assigning the values $a_1, \dots, a_n$ to the coordinate positions $1, \dots, n$ according to some permutation $\pi = (\pi_1, \dots, \pi_n)$, so that $x_i = a_{\pi_i}$. Therefore

$$f(x) = \sum_{i=1}^n f_i(x_i) = \sum_{i=1}^n f_i(a_{\pi_i}).$$

Since E_{nk} is finite and every point corresponds to some permutation $\pi^* = (\pi_1^*, \dots, \pi_n^*)$, minimizing f over E_{nk} is equivalent to minimizing

$$\sum_{i=1}^n f_i(a_{\pi_i})$$

over all permutations $\pi \in P(n)$. ■

Corollary. (*Rearrangement ordering for nondecreasing separable functions*).

Suppose, in addition to the assumptions of Lemma 4.2, that each component function f_i is nondecreasing on the set of admissible coordinate values $\{a_1, \dots, a_n\}$. Then the minimum of $f(x)$ over E_{nk} is attained at a point $x^* = (x_1^*, \dots, x_n^*)$ corresponding to a permutation $(\pi_1, \dots, \pi_n)$ of $\{1, \dots, n\}$ with

$$x_{\pi_i}^* = a_p \quad i = 1, \dots, n,$$

such that

$$f_{\pi_1}(a_1) \geq \dots \geq f_{\pi_n}(a_n). \quad (4.10)$$

Proof. By Lemma 4.2, the minimum is attained at the permutation $\pi^* = (\pi_1^*, \dots, \pi_n^*)$ minimizing

$$\sum_{i=1}^n f_i(a_{\pi_i}).$$

Suppose the optimal assignment π^* violates condition (4.10): there exist positions p and q such that $x_p^* = a_s > a_r = x_q^*$ but $f_p(a_s) < f_q(a_r)$.

Consider the point $\tilde{x}$ obtained by swapping these two coordinate values, that is, $\tilde{x}_p = a_r, \tilde{x}_q = a_s$, with all other coordinates unchanged.

The change in objective value is

$$f(\tilde{x}) - f(x^*) = f_p(a_r) + f_q(a_s) - f_p(a_s) - f_q(a_r).$$

Since f_p and f_q are nondecreasing and $a_s > a_r$, we have $f_p(a_s) \geq f_p(a_r)$ and $f_q(a_s) \geq f_q(a_r)$. Combined with the inversion assumption $f_p(a_s) < f_q(a_r)$, this gives

$$f_q(a_s) \geq f_q(a_r) > f_p(a_s) \geq f_p(a_r),$$

hence $f_p(a_r) + f_q(a_s) < f_p(a_s) + f_q(a_r)$, so $f(\tilde{x}) < f(x^*)$, contradicting the optimality of x^* . Therefore no inversion exists, and condition (4.10) holds. ■

Remark 4.2. In the linear case $f_i(t) = c_i t$, condition (4.10) reduces to

$$c_{\pi_1} a_1 \geq \dots \geq c_{\pi_n} a_n.$$

Since the a_i are ordered nondecreasing, this is equivalent to the coefficient ordering $c_{\pi_1} \geq \dots \geq c_{\pi_n}$ of Lemma 4.1. Thus the Corollary of Lemma 4.2 recovers the linear rearrangement rule as a special case.

The class of separable functions provides a natural extension of the linear case and already covers a wide range of practically relevant objectives. Within this broader functional family, quadratic models occupy a distinguished position due to their transparent geometric interpretation and their close connection with distance-based optimization problems. Rather than analyzing general quadratic forms, we focus on a structurally transparent subclass given by distance-type objectives.

Specifically, we consider the problem of minimizing the Euclidean distance from an external point to the multipermutation point configuration. This formulation admits a direct geometric interpretation in terms of nearest-vertex projection onto the multipermutohedron and provides a natural bridge between permutation extremality and polyhedral distance geometry.

Let $c = (c_1, c_2, \dots, c_n) \in \mathbb{R}^n$ be a fixed vector, and consider the distance function

$$f(x) = \|x - c\|^2. \quad (4.11)$$

We study the problem of minimizing this function over the multipermutation point configuration E_{nk} .

Lemma 4.3. (*Distance minimization to a fixed point*)

Let the function $f(x)$ be given by (4.11). Then

$$\min_{x \in E_{nk}} f(x)$$

is attained at a point x^* obtained by assigning the largest elements of the ordered multiset $a_1 \leq \dots \leq a_n$ to the largest components of the vector c .

Proof. Expanding the squared Euclidean distance yields

$$\|x - c\|^2 = \|x\|^2 - 2 \sum_{i=1}^n c_i x_i + \|c\|^2.$$

For all points $x \in E_{nk}$, the multiset of coordinates is fixed. Hence the quadratic term

$$\|x\|^2 = \sum_{i=1}^n x_i^2 = \sum_{i=1}^n a_i^2$$

is constant on the configuration.

The term $\|c\|^2$ is independent of x as well. Hence the only variable component of the objective is the linear term

$$-2 \sum_{i=1}^n c_i x_i.$$

Therefore the minimization problem

$$\min_{x \in E_{nk}} \|x - c\|^2 \tag{4.12}$$

is equivalent to maximizing the linear function:

$$\max_{x \in E_{nk}} \sum_{i=1}^n c_i x_i$$

By Lemma 4.1, the maximum of this function is attained at the permutation point whose coordinates are ordered oppositely to the coefficients c_i .

Specifically, by Lemma 4.1 applied to the maximization of

$$\sum_{i=1}^n c_i x_i,$$

the maximum is attained by matching the largest a_i with the largest c_i , which is equivalent to reversing the minimization ordering. Consequently, this permutation point minimizes the quadratic distance function on E_{nk} . ■

Remark 4.3. Lemma 4.3 admits a direct geometric interpretation. The multi-permutohedron

$$\Pi_{nk} = \text{conv } E_{nk}$$

is a convex polytope in $\mathbb{R}^n$, and minimizing problem (4.12) is equivalent to finding the vertex of Π_{nk} closest to the point c in the Euclidean metric. In other words, x^* realizes the Euclidean projection of c onto the vertex set of the multi-permutohedron.

This interpretation reveals that extremal properties of distance-type quadratic functions on multipermutation configurations are governed by the polyhedral geometry of the associated multi-permutohedron.

The classes of objective functions analyzed above - linear, separable, and distance-type quadratic - demonstrate that extremal behavior on multipermutation configurations is governed primarily by permutation ordering mechanisms induced by the vertex structure of the carrier.

These results show that many nonlinear optimization problems on MPPC can be reduced, either exactly or approximately, to linear extremal schemes through decomposition, projection, or invariant-based transformations.

At the same time, such reductions rely fundamentally on convexity properties of objective functions and on the existence of convex extensions constructed in §4.1. This observation makes it possible to transfer extremal analysis from special functional classes to the general convex setting.

We therefore proceed to the study of extremal properties of general convex and strongly convex functions defined on multipermutation point configurations.

§4.3. Extremal Properties of General Functions on Multipermutation Point Configurations

In §4.2 we established extremal properties of basic classes of functions on multipermutation configurations, including linear, separable, and distance-type functionals. We now extend these results to arbitrary functions by means of convex extension techniques.

The configuration E_{nk} is vertex-located, and therefore the convex extension framework developed in §4.1 becomes applicable. In particular, for any function f defined on E_{nk} , there exists a convex differentiable extension $\varphi(x)$ defined on a convex set $X \supset E_{nk}$ that coincides with $f(x)$ on the configuration carrier.

This observation makes it possible to transfer extremal analysis from the discrete configuration E_{nk} to the convex domain X while preserving functional values

on the configuration itself. Consequently, optimization problems formulated on E_{nk} may be analyzed through convex-analytic tools applied to $\varphi(x)$.

It should be emphasized that if the original function $f(x)$ is itself convex on a convex set containing E_{nk} , then no continuation is required, and the analysis below applies directly with $\varphi \equiv f$.

To obtain quantitative extremal estimates, let $f(x)$ be a function defined on the multipermutation point configuration E_{nk} , and assume that there exists a convex differentiable extension.

Assume that there exists a convex differentiable extension $\varphi(x)$ defined on a closed convex set $X \supset E_{nk}$ such that $\varphi(x) = f(x)$ for all $x \in E_{nk}$.

The existence of such extensions follows from the continuation framework developed in §4.1. This allows one to estimate extremal values of $f(x)$ on E_{nk} using first-order convexity properties of $\varphi(x)$ on the ambient convex domain X .

Theorem 4.1. (*Gradient estimate of the minimum on E_{nk}*).

Let $f: E_{nk} \rightarrow \mathbb{R}$ be a function defined on the multipermutation point configuration E_{nk} , and let $\varphi: X \rightarrow \mathbb{R}$ be a convex differentiable extension of f to a convex set $X \supseteq E_{nk}$.

Then for every point $x \in X$ the following estimate holds:

$$\min_{y \in E_{nk}} f(y) \geq \varphi(x) + \sum_{i=1}^n \frac{\partial \varphi(x)}{\partial x_{\pi_i}} (a_i - x_{\pi_i}), \quad (4.13)$$

where $(\pi_1, \dots, \pi_n)$ is a permutation of $\{1, \dots, n\}$ satisfying the condition

$$\frac{\partial \varphi(x)}{\partial x_{\pi_1}} \geq \dots \geq \frac{\partial \varphi(x)}{\partial x_{\pi_n}}. \quad (4.14)$$

Proof. Fix an arbitrary point $x \in X$. Since φ is convex and differentiable on the convex set X , the first-order convexity inequality holds for every $y \in X$:

$$\varphi(y) - \varphi(x) \geq \langle \varphi'(x), y - x \rangle.$$

Rewriting the right-hand side gives

$$\varphi(y) \geq \varphi(x) - \langle \varphi'(x), x \rangle + \langle \varphi'(x), y \rangle.$$

Because E_{nk} is contained in X and φ coincides with f on E_{nk} , we obtain, for all $y \in E_{nk}$,

$$f(y) \geq \varphi(x) - \langle \varphi'(x), x \rangle + \langle \varphi'(x), y \rangle.$$

Taking the minimum over $y \in E_{nk}$ yields

$$\min_{y \in E_{nk}} f(y) \geq \varphi(x) - \langle \varphi'(x), x \rangle + \min_{y \in E_{nk}} \langle \varphi'(x), y \rangle$$

The scalar product $\langle \varphi'(x), y \rangle$ is a linear function of y . Hence the problem reduces to minimizing a linear functional over the multipermutation point configuration E_{nk} .

By the extremal property of linear functions on E_{nk} (Lemma 4.1), this minimum is attained at the point whose coordinates form an arrangement of the elements $\{a_1, \dots, a_n\}$ that is monotone with respect to the ordering of the gradient components $\varphi'(x)$.

Therefore,

$$\min_{y \in E_{nk}} \langle \varphi'(x), y \rangle = \sum_{i=1}^n \frac{\partial \varphi(x)}{\partial x_{\pi_i}} a_i,$$

where $(\pi_1, \dots, \pi_n)$ is a permutation of $\{1, \dots, n\}$ satisfying the condition (4.14).

On the other hand,

$$\langle \varphi'(x), x \rangle = \sum_{i=1}^n \frac{\partial \varphi(x)}{\partial x_{\pi_i}} x_{\pi_i}$$

Substituting this expression and the value of the linear minimum into the previous inequality, we obtain

$$\begin{aligned} \min_{y \in E_{nk}} f(y) &\geq \varphi(x) - \langle \varphi'(x), x \rangle + \sum_{i=1}^n \frac{\partial \varphi(x)}{\partial x_{\pi_i}} a_i = \\ &= \varphi(x) + \sum_{i=1}^n \frac{\partial \varphi(x)}{\partial x_{\pi_i}} (a_i - x_{\pi_i}). \end{aligned}$$

This coincides with the estimate stated in the theorem. ■

The gradient estimate obtained above provides a lower bound for the minimum of an arbitrary function on the multipermutation configuration in terms of any convex differentiable extension.

A natural question therefore arises: under what conditions does this estimate become exact at a given configuration point? In other words, when does the supporting linear functional generated by the gradient at a point already realize its minimum on the configuration at that same point?

Since extremal properties of convex functions on vertex-located configurations are governed by the behavior of their linearization, one may expect that optimality on E_{nk} is ensured whenever the gradient ordering is consistent with the permutation structure of the configuration.

The lower estimates obtained above are complemented by the trivial upper bound inherent to discrete minimization. For any point $x \in E_{nk}$ one has

$$\min_{y \in E_{nk}} f(y) \leq f(x).$$

Thus extremal estimates derived via convex continuation acquire a two-sided interpretation: convex extensions generate lower bounds, while function values at

configuration points provide natural upper bounds. The tightness of these bounds depends on the geometric compatibility between the gradient structure of the extension and the permutation ordering of the configuration.

The next theorem formalizes this principle and provides a constructive sufficient condition for optimality expressed entirely in terms of gradient ordering and coordinate arrangement.

Theorem 4.2. (*Sufficient condition for minimum on E_{nk}*).

Let f be a function defined on the multipermutation point configuration E_{nk} , and let φ be a convex differentiable extension of f to a closed convex set $X \supset E_{nk}$, with $\varphi = f$ on E_{nk} . A point $x^* \in E_{nk}$ is a minimizer of f on E_{nk} if there exists a permutation $(\pi_1, \dots, \pi_n)$ of $\{1, \dots, n\}$ such that

$$\frac{\partial \varphi(x^*)}{\partial x_{\pi_1}} \geq \dots \geq \frac{\partial \varphi(x^*)}{\partial x_{\pi_n}} \quad (4.15)$$

and simultaneously

$$x_{\pi_1}^* \leq x_{\pi_2}^* \leq \dots \leq x_{\pi_n}^*. \quad (4.16)$$

Proof. Fix $x^* \in E_{nk}$. Since φ is convex and differentiable on X , for every $y \in X$,

$$\varphi(y) \geq \varphi(x^*) + \langle \varphi'(x^*), y - x^* \rangle.$$

Equivalently,

$$\varphi(y) \geq \varphi(x^*) - \langle \varphi'(x^*), x^* \rangle + \langle \varphi'(x^*), y \rangle.$$

Restricting this inequality to $y \in E_{nk}$ and taking the minimum over E_{nk} gives

$$\min_{y \in E_{nk}} \varphi(y) \geq \varphi(x^*) - \langle \varphi'(x^*), x^* \rangle + \min_{y \in E_{nk}} \langle \varphi'(x^*), y \rangle \quad (4.17)$$

Consider the linear functional $L(y) = \langle \varphi'(x^*), y \rangle$ on E_{nk} .

By Lemma 4.1, its minimum over E_{nk} is attained at the permutation point obtained by matching larger coefficients with smaller multiset values. Condition (4.15) orders the coefficients (the gradient components) in nonincreasing order, and condition (4.16) states that the coordinates of x^* follow the corresponding nondecreasing order. Hence x^* itself attains this minimum, and therefore

$$\min_{y \in E_{nk}} \langle \varphi'(x^*), y \rangle = \langle \varphi'(x^*), x^* \rangle.$$

Substituting into (4.17) yields

$$\min_{y \in E_{nk}} \varphi(y) \geq \varphi(x^*)$$

Since $\varphi = f$ on E_{nk} , we have

$$\min_{x \in E_{nk}} \varphi(x) = \min_{x \in E_{nk}} f(x)$$

and $\varphi(x^*) = f(x^*)$. Hence

$$\min_{y \in E_{nk}} f(y) \geq f(x^*)$$

which implies $f(x^*) \leq f(y)$ for all $y \in E_{nk}$. Thus x^* is a minimizer of f on E_{nk} . ■

Remark 4.4. Conditions (4.15)–(4.16) provide a sufficient condition for x^* to minimize f on E_{nk} ; they are generally not necessary. The reason is that the theorem certifies optimality by exhibiting a particular supporting linear functional, namely the gradient $\langle \varphi'(x^*), y \rangle$, that is minimized at x^* over the discrete set E_{nk} . However, even if x^* is a true minimizer, the minimizing supporting functional at x^* need not coincide with $\varphi'(x^*)$ (and the ordering of gradient components may be non-unique in the presence of ties).

The condition is natural: convexity ensures that minimization of φ over E_{nk} reduces to minimization of the linear functional $\langle \varphi'(x^*), y \rangle$ over E_{nk} . This linear problem on multipermutation sets has an explicit optimal ordering solution, so the requirement that x^* itself follows this ordering (i.e., (4.15)–(4.16)) transparently guarantees optimality among all points in E_{nk} .

The gradient estimates obtained above rely only on convexity of continuation functions and therefore provide linear supporting bounds for the minimum on the configuration. Stronger estimates arise when the extension possesses curvature. We use the standard differential characterization of strong convexity.

A differentiable function φ defined on a convex set X is called ρ -strongly convex if for all $x, y \in X$ the inequality

$$\varphi(y) \geq \varphi(x) + \langle \varphi'(x), y - x \rangle + \rho \|y - x\|^2$$

holds.

In this case, extremal properties are governed not only by linearization but also by quadratic growth away from minimizers.

We therefore proceed to the study of extremal properties generated by strongly convex extensions. As before, the analysis is conducted on convex carriers containing the multipermutation configuration, while extremal conclusions are transferred back to the discrete set E_{nk} .

Theorem 4.3. (Strong convexity minimum estimate on E_{nk}).

Let f be a function defined on the multipermutation point configuration E_{nk} and let φ be a strongly convex extension of f to a closed convex set $X \supset E_{nk}$ with strong convexity parameter $\rho > 0$.

Let

$$x^* = \arg \min_{x \in X} \varphi(x). \quad (4.18)$$

Then the following estimate holds:

$$\min_{y \in E_{nk}} f(y) \geq \varphi(x^*) + \rho \sum_{i=1}^n (x_{\pi_i}^* - a_i)^2$$

where $(\pi_1, \dots, \pi_n)$ is a permutation of $\{1, \dots, n\}$ such that

$$x_{\pi_1}^* \leq \dots \leq x_{\pi_n}^*.$$

Proof. Since φ is strongly convex on X with parameter $\rho > 0$, for all $x, y \in X$ one has

$$\varphi(y) \geq \varphi(x) + \langle \varphi'(x), y - x \rangle + \rho \|y - x\|^2.$$

Because x^* is a minimizer of φ on X , the first-order optimality condition implies

$$\langle \varphi'(x^*), y - x^* \rangle \geq 0 \text{ for all } y \in X.$$

Substituting $x = x^*$ into the strong convexity inequality yields

$$\varphi(y) \geq \varphi(x^*) + \rho \|y - x^*\|^2$$

for all $y \in X$.

Since $E_{nk} \subset X$, the estimate holds in particular for all $y \in E_{nk}$. Taking the minimum over $y \in E_{nk}$ and using the identity $\varphi = f$ on E_{nk} gives

$$\min_{y \in E_{nk}} f(y) \geq \varphi(x^*) + \rho \min_{y \in E_{nk}} \|y - x^*\|^2. \quad (4.19)$$

By Lemma 4.3, the quantity

$$\min_{y \in E_{nk}} \|y - x^*\|^2$$

is attained at the permutation point whose coordinates match the ordered sequence a_i with the ordered coordinates of x^* .

Under ordering (4.14) one obtains

$$\min_{y \in E_{nk}} \|y - x^*\|^2 = \sum_{i=1}^n (x_{\pi_i}^* - a_i)^2$$

Substituting this expression into (4.19) yields the desired estimate. ■

Theorem 4.4. (*Gradient estimate for strongly convex extensions*)

Let $f: E_{nk} \rightarrow \mathbb{R}$ be a function defined on the multipermutation point configuration E_{nk} , and let $\varphi: X \rightarrow \mathbb{R}$ be a ρ -strongly convex differentiable extension of f to a closed convex set $X \supset E_{nk}$ with $\rho > 0$.

Then for every point $x \in X$ the following estimate holds:

$$\min_{y \in E_{nk}} f(y) \geq \varphi(x) - \langle \varphi'(x), x \rangle + \rho \sum_{i=1}^n (x_{\pi_i} - a_i)^2 + \sum_{i=1}^n \frac{\partial \varphi(x)}{\partial x_{\pi_i}} a_i,$$

where $(\pi_1, \dots, \pi_n)$ is an ordering of $\{1, \dots, n\}$ satisfying the condition

$$\frac{\partial \varphi(x)}{\partial x_{\pi_1}} - 2\rho x_{\pi_1} \geq \dots \geq \frac{\partial \varphi(x)}{\partial x_{\pi_n}} - 2\rho x_{\pi_n}. \quad (4.20)$$

Proof. From the strong convexity of the function φ on the set X with parameter $\rho > 0$, for any points $x, y \in X$ the inequality

$$\varphi(y) - \varphi(x) \geq \langle \varphi'(x), y - x \rangle + \rho \|y - x\|^2$$

holds.

Rearranging terms yields

$$\varphi(y) \geq \varphi(x) - \langle \varphi'(x), x \rangle + (\langle \varphi'(x), y \rangle + \rho \|y - x\|^2).$$

Since φ coincides with f on the configuration E_{nk} , this inequality is valid in particular for all $y \in E_{nk}$. Taking the minimum over $y \in E_{nk}$, we obtain

$$\min_{y \in E_{nk}} f(y) \geq \varphi(x) - \langle \varphi'(x), x \rangle + \min_{y \in E_{nk}} (\langle \varphi'(x), y \rangle + \rho \|y - x\|^2).$$

It remains to evaluate the last minimum.

Expanding the quadratic term gives

$$\langle \varphi'(x), y \rangle + \rho \|y - x\|^2 = \langle \varphi'(x) - 2\rho x, y \rangle + \rho \|y\|^2 + \rho \|x\|^2.$$

For all points $x, y \in E_{nk}$ the values

$$\|x\|^2 = \|y\|^2 = \sum_{i=1}^n a_i^2$$

are constant, since all configuration points contain the same multiset of coordinates. Therefore, minimization over E_{nk} reduces to the linear problem

$$\min_{y \in E_{nk}} \langle \varphi'(x) - 2\rho x, y \rangle.$$

This expression represents a linear functional with respect to y . By Lemma 4.1, the minimum of a linear function on the multipermutation configuration E_{nk} is attained at a permutation point whose coordinates form an ordering of the multiset $a_1 \leq a_2 \leq \dots \leq a_n$ consistent with the nonincreasing ordering of the coefficients of the linear form.

Applying this rule to the coefficient vector $\varphi'(x) - 2\rho x$, we obtain a permutation $(\pi_1, \dots, \pi_n)$ satisfying condition (4.20), and

$$\min_{y \in E_{nk}} \langle \varphi'(x) - 2\rho x, y \rangle = \sum_{i=1}^n \left(\frac{\partial \varphi(x)}{\partial x_{\pi_i}} - 2\rho x_{\pi_i} \right) a_i.$$

Substituting this expression into the previous estimate yields

$$\min_{y \in E_{nk}} f(y) \geq \varphi(x) - \langle \varphi'(x), x \rangle + \rho \sum_{i=1}^n (x_{\pi_i} - a_i)^2 + \sum_{i=1}^n \frac{\partial \varphi(x)}{\partial x_{\pi_i}} a_i,$$

which completes the proof. ■

Gradient-based lower estimates and sufficient optimality conditions were derived by transferring convex-analytic arguments from continuous carriers back to the discrete configuration. For strongly convex extensions, curvature effects yielded sharper extremality criteria, combining first-order gradient information with quadratic growth terms.

Since permutation point configurations constitute the nondegenerate special case of multipermutation configurations (unit multiplicities), all results of this paragraph remain valid for E_n without modification.

The analysis will now be extended to partial multipermutation point configurations introduced in §3.4, which generalize full multipermutation carriers by relaxing positional occupancy constraints and are generally not vertex-located.

§4.4. Extremal Properties of Special Functions on Partial Permutation Point Configurations

In §4.3 extremal properties were established for functions defined on full multipermutation point configurations E_{nk} , where the equality

$$E_{nk} = \text{vert conv } E_{nk}$$

permits discrete optimization problems to be reduced directly to linear extremal problems over the convex hull.

The present section addresses partial configurations. As introduced in §3.4, the most general class considered there is the partial multipermutation point configuration

$$\tilde{E}_n^k = \varphi(P_{\eta^k}^n(A)),$$

in which coordinate repetitions are allowed subject to the repetition vector $(n_1, \dots, n_k)$, with $n_1 + \dots + n_k = \eta$, and $n \leq \eta$.

The presence of bounded repetitions substantially complicates exchange arguments and ordering principles. For this reason, attention is restricted to the important special case of configurations without repetitions, namely the subclass

$$E_n^k \subset \tilde{E}_n^k$$

consisting of all n -tuples $x = (x_1, \dots, x_n)$ whose coordinates are pairwise distinct elements selected from a strictly increasing sequence

$$a_1 < \dots < a_k.$$

This corresponds either to the case $n_i = 1$ for all i , or to the deliberate exclusion of configurations that reuse the same numerical value more than once, even when the repetition bounds would permit such reuse.

Every point of E_n^k is obtained by selecting n distinct indices from $\{1, \dots, k\}$ and assigning the corresponding values $a_{l_1}, \dots, a_{l_n}$ to the coordinates in some order. Thus, the set E_n^k is invariant under coordinate permutations and retains structural properties characteristic of permutation-type configurations.

In contrast to the full multipermutation case, the equality

$$E_n^k = \text{vert conv } E_n^k$$

does not generally hold. However, every vertex of the convex hull belongs to the configuration itself:

$$\text{vert conv } E_n^k \subseteq E_n^k.$$

Hence linear minimization over $\text{conv } E_n^k$ reduces to a discrete extremal problem on E_n^k . This permits the application of exchange-based ordering principles analogous to those used for full permutation-type configurations.

The first result establishes the fundamental ordering mechanism for weighted separable objectives defined on E_n^k .

Lemma 4.5. (*Minimum of a weighted separable function with a common generator*).

Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a function and define

$$\Phi(x) = \sum_{i=1}^n c_i h(x_i),$$

where $c_i \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Let $a_1 < \dots < a_k$ be a strictly increasing sequence, and let E_n^k denote the set of all n -component arrangements formed by selecting n pairwise distinct elements from $\{a_1, \dots, a_k\}$.

Let $(\pi_1, \dots, \pi_n)$ be a permutation of $\{1, \dots, n\}$ such that

$$c_{\pi_1} \geq \dots \geq c_{\pi_n}.$$

Let s denote the number of positive coefficients and let $m = n - s$.

Then the minimum of Φ on E_n^k is attained at a configuration

$$x^* = (a_{l_1}, \dots, a_{l_n}),$$

where the index set $\{l_1, \dots, l_n\} \subset \{1, \dots, k\}$ satisfies the following rule:

- indices corresponding to positive coefficients select the s smallest available values of $h(a_j)$;
- indices corresponding to negative coefficients select the m largest available values of $h(a_j)$.

Equivalently, larger coefficients are paired with smaller values of $h(\cdot)$, subject to the admissibility constraints of E_n^k .

Proof. Let $x \in E_n^k$ be arbitrary. Suppose that there exist indices p and q such that $c_p \geq c_q$, but $h(x_p) > h(x_q)$.

Define a new configuration $\tilde{x}$ by exchanging the values in these two coordinates $\tilde{x}_p = x_q$, $\tilde{x}_q = x_p$, with all remaining coordinates unchanged. Since E_n^k is invariant under coordinate permutations, $\tilde{x}$ is admissible.

The change in the objective function equals

$$\Phi(\tilde{x}) - \Phi(x) = (c_p - c_q) \cdot (h(x_q) - h(x_p)).$$

Because $c_p - c_q \geq 0$ and $h(x_q) - h(x_p) < 0$, it follows that

$$\Phi(\tilde{x}) - \Phi(x) \leq 0,$$

with strict inequality whenever both inequalities are strict.

Thus any configuration violating the ordering principle can be improved by such an exchange. Repeating this process finitely many times eliminates all violations. In the resulting configuration, larger coefficients are paired with smaller values of $h(\cdot)$.

Finally, selecting the s smallest values of $h(\cdot)$ for the positive coefficients and the m largest values of $h(\cdot)$ for the negative coefficients produces a configuration x^* that minimizes Φ on E_n^k . ■

Lemma 4.5 covers the general separable case. The linear case, which constitutes the basic extremal component to which more complicated convex functionals will later be reduced, is recorded separately because it admits an explicit closed-form rule.

Lemma 4.6. (*Minimum of a linear function*).

Let

$$L(x) = \sum_{i=1}^n c_i x_i$$

be a linear function on $\mathbb{R}^n$.

Let E_n^k denote the set of all n -component arrangements formed by selecting n pairwise distinct elements from $\{a_1, \dots, a_k\}$.

Let $(\pi_1, \dots, \pi_n)$ be a permutation of $\{1, \dots, n\}$ that reorders the coefficients of $L(x)$ so that

$$c_{\pi_1} \geq c_{\pi_2} \geq \dots \geq c_{\pi_n}.$$

Then the minimum of L on E_n^k (and on $\text{conv } E_n^k$) is attained at a point

$$x^* = (a_{l_1}, \dots, a_{l_n}),$$

where the indices $\{l_1, \dots, l_n\} \subset \{1, \dots, k\}$ are determined by the rule

$$l_j = \begin{cases} j, & \text{if } c_{\pi_j} \geq 0; \\ k-n+j, & \text{if } c_{\pi_j} < 0. \end{cases} \quad (4.21)$$

Proof. This is the specialization of Lemma 4.5 corresponding to $h(t) = t$. Under this choice, the s smallest values of $h(a_j)$ are $a_1, \dots, a_s$, and the m largest are $a_{k-m+1}, \dots, a_k$, which yields exactly rule (4.21). ■

Extremal analysis of convex continuation schemes also requires the treatment of quadratic auxiliary functions. The squared distance to a fixed point provides a canonical example.

Lemma 4.7. (*Distance auxiliary problem: reduction to an assignment problem*)

Let $c \in \mathbb{R}^n$ be fixed and define

$$g(x) = \|x - c\|^2.$$

Let E_n^k denote the set of all n -component arrangements formed by selecting n pairwise distinct elements from the ordered sequence

$$a_1 < \dots < a_k.$$

Then the minimization problem

$$\min_{x \in E_n^k} \|x - c\|^2$$

is equivalent to the combinatorial optimization problem

$$\min \sum_{i=1}^n (a_{l_i} - c_i)^2,$$

where the indices $\{l_1, \dots, l_n\}$ are pairwise distinct from $\{1, \dots, k\}$.

In particular, this is a linear assignment problem with cost matrix

$$d_{ij} = (a_j - c_i)^2.$$

Proof. Expand the objective:

$$\|x - c\|^2 = \sum_{i=1}^n (x_i - c_i)^2 = \sum_{i=1}^n x_i^2 - 2 \sum_{i=1}^n x_i c_i + \|c\|^2.$$

For $x \in E_n^k$ there exists an injection $l: \{1, \dots, n\} \rightarrow \{1, \dots, k\}$ such that $x = (a_{l_1}, \dots, a_{l_n})$.

Hence the minimization reduces to selecting n distinct indices and pairing them with the coordinate positions so as to minimize the total cost

$$\sum_{i=1}^n (a_{l_i} - c_i)^2.$$

The constant term $\|c\|^2$ does not influence the location of the minimum. The resulting problem is precisely the classical linear assignment problem, solvable in polynomial time (for example by the Hungarian method with complexity $O(n^3)$). ■

Remark 4.5. In contrast to the linear case, no closed-form ordering rule generally determines the minimizer. The optimal matching depends on the specific vector c and must be computed by an assignment algorithm. The reduction

remains computationally tractable and will be used in the analysis of convex differentiable extensions.

Remark 4.6. (*Alternative lower bound via rearrangement inequality*). By the rearrangement inequality,

$$\sum_{i=1}^n x_i c_i \leq \sum_{j=1}^n a_{k-j+1} c_{\pi_j},$$

and since the elements $a_1 < \dots < a_k$ are of arbitrary sign, let $s_1, \dots, s_n$ denote the indices of the n elements with smallest absolute values a_j . Then

$$\sum_{i=1}^n x_i^2 \geq \sum_{i=1}^n a_{s_i}^2.$$

Substituting these bounds into the expansion of $\|x - c\|^2$ yields the lower estimate

$$\min_{x \in E_n^k} \|x - c\|^2 \geq \sum_{i=1}^n a_{s_i}^2 + \|c\|^2 - 2 \sum_{j=1}^n a_{k-j+1} c_{\pi_j},$$

which follows the same ordering principle as in Lemmas 4.5–4.6 and is useful for theoretical estimates when solving the assignment problem exactly is undesirable.

The extremal properties established in Lemmas 4.5–4.7, including ordering rules for linear and separable objectives and the reduction of distance minimization to an assignment problem, form the combinatorial foundation for the convex extension analysis. In §4.5 these results are used to derive explicit lower bounds and sufficient optimality conditions for convex differentiable functions on E_n^k .

§4.5. Lower Bounds and Optimality Conditions for Convex Extensions on Partial Permutation Point Configurations

In §4.4 we established the extremal structure of linear and distance-type functionals on the configuration E_n^k . In particular, we showed that minimization problems on this discrete set reduce to ordered matching between coefficients and admissible coordinate values. We now turn to the study of convex differentiable extensions defined on a closed convex set X containing E_n^k . The key idea is that convexity allows one to compare function values through supporting linear functionals, while the extremal results of Lemmas 4.5–4.7 determine the precise behavior of these linear functionals on E_n^k . This approach naturally leads to lower bounds for convex functions on E_n^k and, subsequently, to sufficient optimality conditions.

Recall that each point of E_n^k has the form $x = (a_{l_1}, \dots, a_{l_n})$ where $\{l_1, \dots, l_n\} \subset \{1, \dots, k\}$ is a set of n distinct indices. The minimization of a linear function over E_n^k therefore involves both the selection of n values from $(a_1, \dots, a_k)$ and their assignment to coordinate positions.

By Lemma 4.6 and rule (4.21), the optimal index set and coordinate ordering are determined simultaneously by the nonincreasing ordering of the coefficients: indices corresponding to non-negative coefficients select the s smallest available values, and indices corresponding to negative coefficients select the m largest. With this in mind, the minimum of any linear function

$$L(x) = \sum_{i=1}^n c_i x_i$$

over E_n^k can be written explicitly as

$$\min_{x \in E_n^k} \sum_{j=1}^n c_{\pi_j} a_{l_j},$$

where $(\pi_1, \dots, \pi_n)$ is the permutation $\{1, \dots, n\}$ with

$$c_{\pi_1} \geq \dots \geq c_{\pi_n},$$

and $\{l_1, \dots, l_n\}$ is the index set given by rule (4.21).

Theorem 4.5. (*Lower estimate for convex differentiable extensions*). Let φ be convex and differentiable on a closed convex set X such that $E_n^k \subseteq X$. Then for every $x \in X$,

$$\min_{y \in E_n^k} \varphi(y) \geq \varphi(x) + \sum_{j=1}^n \frac{\partial \varphi(x)}{\partial x_{\pi_j}} (a_{l_j} - x_{\pi_j}), \quad (4.22)$$

where $(\pi_1, \dots, \pi_n)$ is the permutation $\{1, \dots, n\}$ satisfying

$$\frac{\partial \varphi(x)}{\partial x_{\pi_1}} \geq \dots \geq \frac{\partial \varphi(x)}{\partial x_{\pi_n}},$$

and $\{l_1, \dots, l_n\} \subset \{1, \dots, k\}$ is determined by rule (4.21) applied to these gradient components.

Proof. Fix an arbitrary point $x \in X$. Since φ is convex and differentiable on X , the first-order supporting inequality holds for any $y \in X$:

$$\varphi(y) \geq \varphi(x) + \langle \varphi'(x), y - x \rangle.$$

Rewriting the right-hand side gives

$$\varphi(y) \geq \varphi(x) - \langle \varphi'(x), x \rangle + \langle \varphi'(x), y \rangle.$$

Since $E_n^k \subset X$, restricting to $y \in E_n^k$ and taking the minimum over E_n^k yields

$$\min_{y \in E_n^k} \varphi(y) \geq \varphi(x) - \langle \varphi'(x), x \rangle + \min_{y \in E_n^k} \langle \varphi'(x), y \rangle. \quad (4.23)$$

The last term is the minimum of a linear function over E_n^k . As established in the preceding paragraph, we have

$$\min_{y \in E_n^k} \langle \varphi'(x), y \rangle = \sum_{j=1}^n \frac{\partial \varphi(x)}{\partial x_{\pi_j}} a_{\pi_j}$$

On the other hand,

$$\langle \varphi'(x), x \rangle = \sum_{j=1}^n \frac{\partial \varphi(x)}{\partial x_{\pi_j}} x_{\pi_j}.$$

Substituting both expressions and combining terms gives estimate (4.22). ■

The lower estimate of Theorem 4.5 relies only on the convexity of φ and therefore uses a linear supporting functional at an arbitrary point $x \in X$. It provides a simple and computationally attractive way to obtain a lower bound for the discrete minimum without solving the original problem directly. However, if the extension possesses curvature (i.e., is strongly convex), the supporting inequality becomes quadratic. This allows one to control the discrete minimum not only by a linear extremal problem, but also by the distance from the continuous minimizer to the configuration E_n^k .

Theorem 4.6. (*Sufficient condition for optimality*).

In order that a point $x^0 \in E_n^k$ be a minimizer of φ on E_n^k , it is sufficient that the gradient components of φ at x^0 satisfy

$$\frac{\partial \varphi(x^0)}{\partial x_{\pi_1}} \geq \dots \geq \frac{\partial \varphi(x^0)}{\partial x_{\pi_n}},$$

and that the coordinates of x^0 themselves satisfy the matching rule (4.21).

Proof. From Theorem 4.5 we have estimate (4.23) for any $x \in X$. Taking $x = x^0$ and using Lemma 4.6, the minimum of the linear functional $\langle \varphi'(x^0), y \rangle$ over E_n^k is attained at a configuration whose coordinates satisfy rule (4.21) with respect to the ordering of the components of $\varphi'(x^0)$. By the assumption of the theorem the point x^0 itself satisfies this rule. Therefore

$$\langle \varphi'(x^0), x^0 \rangle = \min_{y \in E_n^k} \langle \varphi'(x^0), y \rangle.$$

Substituting this equality into the inequality (4.23) yields

$$\min_{y \in E_n^k} \varphi(y) \geq \varphi(x^0)$$

Since $x^0 \in E_n^k$, the reverse inequality

$$\min_{y \in E_n^k} \varphi(y) \leq \varphi(x^0)$$

is obvious. Hence equality holds and x^0 is a minimizer of φ on E_n^k . ■

The estimate obtained in Theorem 4.5 is based solely on convexity. A natural question arises: what additional information can be extracted if the function is not only convex but also strongly convex? In this case the supporting inequality gains a quadratic term, which opens the way to controlling the discrete minimum via the Euclidean distance to the continuous minimizer.

Theorem 4.7. (*Strongly convex function: lower bound for the minimum*).

Let φ be strongly convex with parameter $\rho > 0$ on a closed convex set X such that $E_n^k \subset X$. Denote by y^* the unique minimizer of φ over X . Then

$$\min_{x \in E_n^k} \varphi(x) \geq \varphi(y^*) + \rho \min_{x \in E_n^k} \|x - y^*\|^2. \quad (4.24)$$

Proof. By the definition of strong convexity with parameter $\rho > 0$, for every $x \in X$ the inequality

$$\varphi(x) - \varphi(y^*) \geq \rho \|x - y^*\|^2$$

holds.

Restricting this inequality to the discrete set $E_n^k \subset X$ and taking the minimum over the left-hand side, we immediately obtain the required estimate (4.24). The minimum on the right-hand side is attained at the point $x^* \in E_n^k$ closest to y^* in the Euclidean norm. ■

Theorem 4.7 gives a sharp lower bound expressed in terms of the distance from the continuous minimizer y^* to the discrete set E_n^k . This bound is theoretically elegant, but its practical use requires solving the nearest-point problem to y^* . If, however, φ is not only strongly convex but also differentiable, one can derive a stronger quadratic supporting inequality at an arbitrary point $x \in X$. After completing the square, the resulting bound again reduces to a nearest-point problem, but now to a shifted center that depends on both x and the gradient $\varphi'(x)$.

Theorem 4.8. (*Gradient-based bound for a strongly convex differentiable function*).

Let φ be strongly convex with parameter $\rho > 0$ and differentiable on a closed convex set $X \supset E_n^k$. Then for any $x \in X$,

$$\begin{aligned} \min_{y \in E_n^k} \varphi(y) &\geq \varphi(x) - \langle \varphi'(x), x \rangle + \rho \|x\|^2 - \\ &\quad - \rho \min_{y \in E_n^k} \|y - c\|^2 + \left(\frac{1}{4\rho} \right) \|\varphi'(x)\|^2, \end{aligned} \quad (4.25)$$

where the vector c is defined by

$$c = x - \frac{1}{2\rho} \varphi'(x) \quad (4.26)$$

Proof. Since φ is strongly convex with parameter $\rho > 0$ and differentiable on X , for any $x, y \in X$ the following inequality holds:

$$\varphi(y) - \varphi(x) \geq \langle \varphi'(x), y - x \rangle + \rho \|y - x\|^2.$$

Hence, for $y \in E_n^k$, we have

$$\begin{aligned} \varphi(y) &\geq \varphi(x) + \langle \varphi'(x), y - x \rangle + \rho \|y - x\|^2 = \\ &= \varphi(x) - \langle \varphi'(x), x \rangle + \langle \varphi'(x), y \rangle + \rho \|y - x\|^2 \end{aligned}$$

Taking the minimum over $y \in E_n^k$ gives

$$\begin{aligned} \min_{y \in E_n^k} \varphi(y) &\geq \varphi(x) - \langle \varphi'(x), x \rangle + \\ &+ \min_{y \in E_n^k} (\langle \varphi'(x), y \rangle + \rho \|y - x\|^2) \end{aligned}$$

Taking into account the expand the squared norm $\|y - x\|^2$, we

$$\begin{aligned} \langle \varphi'(x), y \rangle + \rho \|y - x\|^2 &= \langle \varphi'(x), y \rangle + \rho \|y\|^2 - 2\rho \langle x, y \rangle + \rho \|x\|^2 = \\ &= \rho \|y\|^2 + \langle \varphi'(x) - 2\rho x, y \rangle + \rho \|x\|^2. \end{aligned}$$

Introduce the auxiliary vector

$$c = x - (1/2\rho) \varphi'(x).$$

A direct computation shows that

$$\begin{aligned} \rho \|y\|^2 + \langle \varphi'(x) - 2\rho x, y \rangle &= \\ = \rho \|y - c\|^2 - (1/4\rho) \|\varphi'(x)\|^2. \end{aligned}$$

Therefore

$$\begin{aligned} \langle \varphi'(x), y \rangle + \rho \|y - x\|^2 &= \\ = \rho \|y - c\|^2 - (1/4\rho) \|\varphi'(x)\|^2 + \rho \|x\|^2. \end{aligned}$$

Substituting this identity into the previous inequality, we arrive at the desired estimate (4.25). The point y^* minimizing $\|y - c\|^2$ over E_n^k is the solution of the corresponding auxiliary nearest-point problem. ■

Theorem 4.8 gives, for any $x \in X$, a lower bound expressed through an auxiliary nearest-point problem to the shifted center c . If solving this auxiliary problem exactly appears computationally expensive, one may use a simpler (though generally non-sharp) lower bound based on the rearrangement inequality and the choice of elements with smallest absolute values.

Remark 4.7. If solving the auxiliary nearest-point problem

$$\min_{y \in E_n^k} \|y - c\|^2$$

exactly is computationally expensive, one may use the following easily computable (though generally non-sharp) lower bound:

$$\min_{y \in E_n^k} \|y - c\|^2 \geq \sum_{j=1}^n a_{s_j}^2 + \sum_{j=1}^n c_{\pi_j}^2 - 2 \sum_{j=1}^n a_{s_j} c_{\pi_j},$$

where $\{s_1, \dots, s_n\}$ are the indices of the n elements with the smallest absolute values from $\{|a_1|, \dots, |a_n|\}$, and $\{\pi_1, \dots, \pi_n\}$ reorder the components of c so that $c_{\pi_1} \geq \dots \geq c_{\pi_n}$.

This estimate follows directly from the rearrangement inequality applied to the cross term and the choice of the n smallest squares for the quadratic part. It preserves the same ordering principle as Lemmas 4.5–4.6 and can be evaluated almost instantly.

Theorem 4.9. (*Weak gradient-based lower bound*)

Let φ be strongly convex with parameter $\rho > 0$ and differentiable on a closed convex set $X \supset E_n^k$. Then for any $x \in X$

$$\begin{aligned} \min_{y \in E_n^k} \varphi(y) &\geq \varphi(x) - \langle \varphi'(x), x \rangle + \rho \|x\|^2 + \\ &+ \rho \sum_{j=1}^n a_{s_j}^2 + \min_{y \in E_n^k} \langle \varphi'(x) - 2\rho x, y \rangle. \end{aligned} \quad (4.27)$$

Proof. Starting from the inequality derived in the proof of Theorem 4.8, we have

$$\begin{aligned} \min_{y \in E_n^k} \varphi(y) &\geq \varphi(x) - \langle \varphi'(x), x \rangle + \\ &+ \min_{y \in E_n^k} (\langle \varphi'(x), y \rangle + \rho \|y - x\|^2) = \\ &= \varphi(x) - \langle \varphi'(x), x \rangle + \rho \|x\|^2 + \min_{y \in E_n^k} (\langle \varphi'(x) - 2\rho x, y \rangle + \rho \|y\|^2). \end{aligned}$$

The quadratic term $\rho \|y\|^2$ is minimized by choosing the n elements with the smallest absolute values (indices s_j). For the remaining linear term $\langle \varphi'(x) - 2\rho x, y \rangle$ we apply Lemma 4.6. Its minimum is attained at the configuration given by rule (4.21) with respect to the ordering of the components of the vector $\varphi'(x) - 2\rho x$. Substituting these two lower bounds yields the desired weak estimate (4.27). Although this bound is weaker than the exact one of Theorem 4.8, it requires only sorting operations and the application of rule (4.21), without solving any assignment problem. ■

Theorem 4.10. (*Sufficient condition for a minimum in the strongly convex differentiable case*).

Let φ be strongly convex with parameter $\rho > 0$ and differentiable on a closed convex set $X \supset E_n^k$. A point $x^* \in E_n^k$ is a minimizer of φ on E_n^k if it satisfies

$$x^* = \arg \min_{y \in E_n^k} \|y - x^* + (1/2\rho)\varphi'(x^*)\|^2. \quad (4.28)$$

Proof. Substitute $x = x^*$ into the inequality of Theorem 4.8. Then

$$c^* = x^* - (1/2\rho) \varphi'(x^*).$$

By the assumption of the theorem, x^* attains the minimum in the auxiliary nearest-point problem, i.e.,

$$\|x^* - c^*\|^2 = \min_{y \in E_n^k} \|y - c^*\|^2.$$

Substituting this equality into (4.25) makes the right-hand side equal to $\varphi(x^*)$. Thus

$$\min_{y \in E_n^k} \varphi(y) \geq \varphi(x^*).$$

The reverse inequality is trivial since $x^* \in E_n^k$. Therefore equality holds and x^* is a minimizer of φ on E_n^k . ■

The results obtained in this section provide a complete set of lower bounds and sufficient optimality conditions for convex and strongly convex extensions on the configuration E_n^k . These estimates are expressed through linear matching (Theorems 4.5–4.6), through the distance to the continuous minimizer (Theorem 4.7), or through auxiliary nearest-point problems to a shifted center (Theorems 4.8 and 4.10). The weak estimate of Theorem 4.9 and Remark 4.7 offer computationally inexpensive alternatives when solving the assignment problem exactly is undesirable.

All the results of this section were derived under the assumption that coordinate repetitions are absent, that is, for configurations E_n^k . It is natural to ask to what extent these results carry over to the more general partial multipermutation configurations $\tilde{E}_n^k$, which allow bounded repetitions subject to the vector $(\eta_1, \dots, \eta_k)$. Several features remain valid without modification: the vertex property $\text{vert conv } \tilde{E}_n^k \subseteq \tilde{E}_n^k$ is preserved, so linear minimization over the convex hull continues to yield extremal points in the discrete set, and the lower bounds based on linear supporting functions (Theorems 4.5–4.6) carry over directly. However, exchange arguments, which form the basis of Lemmas 4.5–4.6, no longer apply automatically, since swapping coordinates may violate multiplicity constraints. The ordering rule (4.21) remains valid only until multiplicity limits are reached; in general, one must solve a bounded assignment or transportation problem. Quadratic auxiliary minimization likewise becomes a nearest-point problem with repetition constraints, requiring more involved computational techniques.

Thus, while the overall structure of the estimates survives the passage to partial multipermutation configurations, both the analysis and the computation become substantially more demanding. The ordering rule (4.21) remains valid only until multiplicity limits are reached; beyond that threshold, bounded assignment or transportation problems must be solved, and quadratic auxiliary minimization requires nearest-point techniques with repetition constraints.

§4.6. Extremal Properties of Convex Functions on Unbounded Partial Multipermutation Point Configurations

We now consider the limiting case in the hierarchy of combinatorial point configurations – the unbounded partial multipermutation point configurations. As defined in §3.7, these configurations arise when all multiplicity constraints are removed while the configuration length n remains fixed. Equivalently, they are the Cartesian product sets

$$B_k^n = G^n \subset \mathbb{R}^n,$$

where $G = \{g_1 < g_2 < \dots < g_k\}$ is the finite numerical realization. Every point $x = (x_1, \dots, x_n) \in B_k^n$ is formed by choosing each coordinate x_i independently from the set G . This complete independence of coordinates is the decisive structural feature that radically simplifies all extremal problems compared to the bounded cases studied earlier: any separable objective function completely decouples into a sum of n independent one-dimensional subproblems, exchange arguments operate without any restrictions, and many auxiliary optimization tasks admit explicit closed-form solutions.

We begin with the most general class of separable functions.

Lemma 4.11. (*Minimum of a weighted separable function*).

Let

$$\Phi(x) = \sum_{i=1}^n c_i h_i(x_i),$$

where $c_i \in \mathbb{R}$ and each $h_i: \mathbb{R} \rightarrow \mathbb{R}$ for $i = 1, \dots, n$. The minimum of Φ on B_k^n is attained at the point $x^* = (x_1^*, \dots, x_n^*)$ in which, for each coordinate i independently, x_i^* minimizes h_i over G when $c_i \geq 0$, x_i^* maximizes h_i over G when $c_i < 0$.

Proof. The set B_k^n consists of all possible n -tuples whose coordinates are chosen independently from G . Therefore any point $x \in B_k^n$ can be written as $x = (x_1, \dots, x_n)$ with each $x_i \in G$.

Because of this independence, the objective function Φ separates completely into a sum of n independent terms:

$$\Phi(x) = \sum_{i=1}^n c_i h_i(x_i).$$

The minimum of a sum of independent terms equals the sum of the minima of each term taken separately. Hence it is sufficient to minimize each summand $c_i h_i(x_i)$ over the set G independently of the other coordinates.

Fix an arbitrary index i . Consider the term $c_i h_i(x_i)$.

- If $c_i \geq 0$, the coefficient is nonnegative. The product $c_i h_i(x_i)$ attains its smallest value precisely when $h_i(x_i)$ is as small as possible. Therefore we choose x_i^* to be any element of G at which h_i reaches its minimum.
- If $c_i < 0$, the coefficient is negative. In this case the product $c_i h_i(x_i)$ attains its smallest value when $c_i h_i(x_i)$ is as large as possible. Therefore we choose x_i^* to be any element of G at which h_i reaches its maximum.

Since the choice of the optimal value for coordinate i does not restrict the admissible choices for the remaining coordinates, the vector $x^* = (x_1^*, \dots, x_n^*)$ constructed by performing this independent minimization for every i is an admissible point of B_k^n . Moreover, by the separability of Φ , this point achieves the global minimum of the function on the entire set B_k^n . ■

Lemma 4.12. (*Minimum of a linear function*)

Let

$$L(x) = \sum_{i=1}^n c_i x_i.$$

The minimum of L on B_k^n is attained at the point $x^* = (x_1^*, \dots, x_n^*)$ in which, for each coordinate i independently,

$$x_i^* = \begin{cases} g_1 & \text{when } c_i \geq 0; \\ g_k & \text{when } c_i < 0. \end{cases} \quad (4.29)$$

Proof. We apply the result of Lemma 4.11 with the special choice $h_i(t) = t$ for every i . In this case each term of the objective becomes $c_i x_i$.

Fix an arbitrary index i .

- If $c_i \geq 0$, the coefficient is nonnegative, so $c_i x_i$ is minimized when x_i is as small as possible. The smallest admissible value in G is g_1 .
- If $c_i < 0$, the coefficient is negative, so $c_i x_i$ is minimized when x_i is as large as possible. The largest admissible value in G is g_k .

Performing this choice independently for each coordinate i yields a point x^* that belongs to B_k^n . Since the objective $L(x)$ is the sum of independent terms $c_i x_i$ and each term attains its individual minimum at the chosen value, the sum $L(x^*)$ is the global minimum of L on the entire set B_k^n . ■

Lemma 4.13. (*Minimum of the squared distance*)

Let $g(x) = \|x - c\|^2$ with fixed $c \in \mathbb{R}^n$. The minimum of g on B_k^n is attained at the point $x^* = (x_1^*, \dots, x_n^*)$ in which, for each coordinate i independently,

$$x_i^* = \begin{cases} g_1 & \text{when } c_i \leq g_1, \\ g_k & \text{when } c_i \geq g_k, \\ \arg \min_{g_j \in G} |g_j - c_i| & \text{otherwise.} \end{cases} \quad (4.30)$$

Proof. The squared Euclidean distance separates completely into a sum of n independent terms:

$$\|x - c\|^2 = \sum_{i=1}^n (x_i - c_i)^2.$$

Each term $(x_i - c_i)^2$ depends only on the choice of $x_i \in G$ and must be minimized separately. The value of x_i that minimizes $(x_i - c_i)^2$ is the element of G that is closest to c_i in the ordinary Euclidean metric.

- If $c_i \leq g_1$, then the entire set G lies to the right of c_i , so the closest point is the leftmost element g_1 .
- If $c_i \geq g_k$, then the entire set G lies to the left of c_i , so the closest point is the rightmost element g_k .
- In the remaining case ($g_1 < c_i < g_k$), the closest point is any g_j for which the distance $|g_j - c_i|$ attains its minimum over G .

Because the choice for one coordinate does not affect the admissibility of the choices for the other coordinates, the vector x^* constructed by performing this independent nearest-point selection for every i belongs to B_k^n and achieves the global minimum of g on the entire set.

Theorem 4.11. (*Lower estimate for convex differentiable extensions*) Let φ be convex and differentiable on a closed convex set X such that $B_k^n \subset X$. Then for every $x \in X$,

$$\min_{y \in B_k^n} \varphi(y) \geq \varphi(x) - \langle \varphi'(x), x \rangle + \sum_{i=1}^n \frac{\partial \varphi(x)}{\partial x_i} x_i^*, \quad (4.31)$$

where $x^* = (x_1^*, \dots, x_n^*)$ is the point of B_k^n determined coordinatewise by rule (4.29).

Proof. By convexity and differentiability of φ on X , the supporting inequality

$$\varphi(y) \geq \varphi(x) + \langle \varphi'(x), y - x \rangle$$

holds for any $y \in X$. Rewriting the right-hand side gives

$$\varphi(y) \geq \varphi(x) - \langle \varphi'(x), x \rangle + \langle \varphi'(x), y \rangle.$$

Restricting y to B_k^n and taking the minimum yields

$$\min_{y \in B_k^n} \varphi(y) \geq \varphi(x) - \langle \varphi'(x), x \rangle + \min_{y \in B_k^n} \langle \varphi'(x), y \rangle.$$

The last term is the minimum of the linear function $\langle \varphi'(x), y \rangle$ on B_k^n , which is given explicitly by rule (4.29) of Lemma 4.12. Substituting this value completes the proof. ■

Theorem 4.12. (*Sufficient condition for optimality*)

Let $x^0 \in B_k^n$. If, for every $i = 1, \dots, n$, the gradient components of φ at x^0 satisfy the following conditions:

$$\frac{\partial \varphi(x^0)}{\partial x_i} \begin{cases} \geq 0 & \text{whenever } x_i^0 = g_1, \\ < 0 & \text{whenever } x_i^0 = g_k, \\ \text{is arbitrary} & \text{whenever } g_1 < x_i^0 < g_k, \end{cases} \quad (4.32)$$

then x^0 is a minimizer of φ on B_k^n .

Proof. From Theorem 4.11 we have for any $x \in X$

$$\min_{y \in B_k^n} \varphi(y) \geq \varphi(x) - \langle \varphi'(x), x \rangle + \min_{y \in B_k^n} \langle \varphi'(x), y \rangle.$$

Setting $x = x^0$ and using that x^0 itself attains the minimum of the linear functional $\langle \varphi'(x^0), y \rangle$, we obtain

Setting $x = x^0$, it remains to verify that x^0 itself attains the minimum of the linear functional $\langle \varphi'(x^0), y \rangle$ over B_k^n . By Lemma 4.12, this minimum is attained coordinatewise: for each i , the optimal value is g_1 when the coefficient

$$\frac{\partial \varphi(x^0)}{\partial x_i} \geq 0,$$

and g_k when it is negative. Condition (4.32) states precisely that x^0 satisfies this coordinatewise rule, so x^0 minimizes $\langle \varphi'(x^0), y \rangle$ over B_k^n . Therefore:

$$\min_{y \in B_k^n} \varphi(y) \geq \varphi(x^0).$$

Since x^0 belongs to B_k^n , the minimum of φ over this set cannot exceed the value of the function at x^0 . Therefore

$$\min_{y \in B_k^n} \varphi(y) \leq \varphi(x^0).$$

Comparing both inequalities gives equality, so x^0 is a minimizer of φ on B_k^n . ■

Theorem 4.13. (Strongly convex function: lower bound for the minimum)

Let φ be strongly convex with parameter $\rho > 0$ on a closed convex set X such that $B_k^n \subset X$. Let y^* be the unique minimizer of φ over X . Then

$$\min_{x \in B_k^n} \varphi(x) \geq \varphi(y^*) + \rho \min_{x \in B_k^n} \|x - y^*\|^2$$

Proof. By the definition of strong convexity with parameter $\rho > 0$, for every $x \in X$ the inequality

$$\varphi(x) - \varphi(y^*) \geq \rho \|x - y^*\|^2$$

holds. Restricting this inequality to $x \in B_k^n$ and taking the minimum on the left-hand side immediately yields the desired estimate. ■

Theorem 4.14. (*Gradient-based bound for a strongly convex differentiable function*)

Let φ be strongly convex with parameter $\rho > 0$ and differentiable on a closed convex set $X \supset B_k^n$. Then for any $x \in X$,

$$\begin{aligned} \min_{y \in B_k^n} \varphi(y) &\geq \varphi(x) - \langle \varphi'(x), x \rangle + \rho \|x\|^2 - \\ &\quad - \frac{1}{4\rho} \|\varphi'(x)\|^2 + \rho \min_{y \in B_k^n} \|y - c\|^2, \end{aligned} \quad (4.33)$$

where $c = x - (1/2\rho) \varphi'(x)$.

Proof. Strong convexity and differentiability yield

$$\varphi(y) - \varphi(x) \geq \langle \varphi'(x), y - x \rangle + \rho \|y - x\|^2.$$

Hence

$$\min_{y \in B_k^n} \varphi(y) \geq \varphi(x) - \langle \varphi'(x), x \rangle + \min_{y \in B_k^n} (\langle \varphi'(x), y \rangle + \rho \|y - x\|^2).$$

Expanding the squared norm and completing the square leads to the auxiliary vector

$$c = x - \frac{1}{2\rho} \varphi'(x)$$

and the required identity. Substituting this identity and taking the minimum over $y \in B_k^n$ gives the desired bound.

Theorem 4.15. (*Sufficient condition for a minimum in the strongly convex differentiable case*).

Let φ be strongly convex with parameter $\rho > 0$ and differentiable on a closed convex set $X \supset B_k^n$. Let $x^* \in B_k^n$. If

$$x^* = \arg \min_{y \in B_k^n} \left\| y - x^* + \frac{1}{2\rho} \varphi'(x^*) \right\|^2, \quad (4.34)$$

then x^* is a minimizer of φ on B_k^n .

Proof. Substitute $x = x^*$ into the inequality of Theorem 4.14. Let

$$c^* = x^* - \frac{1}{2\rho} \varphi'(x^*).$$

By assumption x^* attains the minimum in the auxiliary problem, i.e.,

$$\|x^* - c^*\|^2 = \min_{y \in B_k^n} \|y - c^*\|^2.$$

Substituting this equality makes the right-hand side equal to $\varphi(x^*)$. Since x^* belongs to B_k^n , the minimum of φ over this set cannot exceed the value of the function at x^* . Therefore

$$\min_{y \in B_k^n} \varphi(y) \leq \varphi(x^*).$$

Comparing both inequalities gives equality, so x^* is a minimizer. ■

The unbounded case reveals the ultimate simplification of the theory developed in this chapter. Because the coordinates are completely independent, all exchange arguments operate without any restriction, all ordering rules reduce to independent per-coordinate choices, and every auxiliary nearest-point problem admits an explicit closed-form solution. These features make unbounded partial multipermutation configurations not only the simplest, but also the most computationally attractive class in the entire hierarchy. They serve as a natural reference model against which the increased complexity of bounded and partial configurations can be clearly seen and measured.

§4.7. Convex Extensions on Special Partial Point Configurations

In §4.1 we showed that for full multipermutation point configurations E_{nk} a convex (and even strongly convex) extension always exists and can be constructed explicitly. The key reason was the vertex-located property $E_{nk} = \text{vert conv } E_{nk}$, which allowed us to apply the general existence theorems from Chapter 2 and then develop concrete analytic schemes for polynomials, quadratic functions, and strongly convex augmentations.

We now address the more general situation of partial multipermutation point configurations $\tilde{E}_n^k$ (with bounded repetitions) and unbounded configurations B_k^n . In these cases the vertex-located property no longer holds in general. Consequently, the existence of convex extensions to any convex superset $X \supset \tilde{E}_n^k$ or $X \supset B_k^n$ are no longer guaranteed for arbitrary functions prescribed on the discrete set. However, there exist two important special classes in which the configuration remains vertex-located and therefore admits a convex continuation. These two classes are the partial permutation point configuration E_n^{n+1} (selection of n from distinct values from $n + 1$ available) and the unbounded configuration B_k^n when $k = 2$ (the Boolean hypercube case).

We now present explicit constructive schemes for convex extensions in each of these cases, following the same methodological spirit as in §4.1: first for

polynomial functions, then for quadratic objectives, and finally for strongly convex extensions. All constructions are given in full detail with complete proofs of convexity.

Let the underlying set consist of n distinct values $a_1 < \dots < a_{n+1}$. The configuration E_n^{n+1} consists of all n -tuples formed by selecting n pairwise distinct elements from $\{a_1, \dots, a_{n+1}\}$.

By Theorem 3.18 and Remark 3.6 of §3.6, this configuration is vertex-located:.

$$E_n^{n+1} = \text{vert conv } E_n^{n+1}.$$

Moreover, Remark 3.6 establishes an explicit bijection between E_n^{n+1} and the full permutation point configuration E_{n+1}^{n+1} via the affine coordinate relation, which we now exploit for the construction of convex extensions

To construct an explicit convex extension, we take into account that

$$E_{n+1}^{n+1} = E_{n+1, n+1}$$

and use the projection of $E_{n+1}^{n+1} \subset \mathbb{R}^{n+1}$ on $\mathbb{R}^n$.

Consider the auxiliary full permutation point configuration E_{n+1}^{n+1} generated by the same values $\{a_1, \dots, a_{n+1}\}$. On E_{n+1}^{n+1} we define an auxiliary function $\psi(x_1, \dots, x_{n+1})$ by setting

$$\psi(x_1, \dots, x_{n+1}) = \varphi(x_1, \dots, x_n),$$

where φ is the original function defined on E_n^{n+1} .

This definition is consistent because on E_{n+1}^{n+1} the last coordinate x_{n+1} is determined by the first n via the affine relation

$$x_{n+1} = \sum_{i=1}^{n+1} a_i - \sum_{i=1}^n x_i,$$

so ψ depends only on $(x_1, \dots, x_n)$.

The resulting function of the first n variables is defined on the projection of E_{n+1}^{n+1} onto the first n coordinates, which coincides exactly with E_n^{n+1} .

Because the full permutation configuration E_{n+1}^{n+1} is vertex-located and possesses a convex extension (as constructed in §4.1), the function obtained after projection inherits the convexity property on $\mathbb{R}^n$. Consequently, this approach yields an explicit convex continuation of the original function φ to any convex superset containing E_n^{n+1} . In addition, if the extension constructed on E_{n+1}^{n+1} is strongly convex with parameter $\rho > 0$, then augmenting it by the quadratic correction term

$$\rho \left(\sum_{i=1}^{n+1} x_i - \sum_{i=1}^{n+1} a_i \right)^2$$

results in a strongly convex extension also for the partial permutation configuration E_n^{n+1} .

For the unbounded configuration $B_n^k = G^n$ the situation is different. Here every coordinate varies independently over G , so the set is the full Cartesian product. Geometrically this means that B_n^k coincides with the set of all points whose coordinates are chosen from G . The convex hull is the hyperrectangle

$$\Pi_n^\infty = \text{conv } B_n^k = \prod_{i=1}^n [g_1, g_k].$$

For general $k > 2$ the set B_n^k is not vertex-located. The vertices of the hyperrectangle Π_n^∞ are exactly the 2^n points where each coordinate equals either g_1 or g_k . All other points of B_n^k (those with at least one coordinate strictly between g_1 and g_k) lie in the interior of faces, edges, or the volume of the hyperrectangle. Therefore

$$B_n^k \neq \text{vert conv } B_n^k$$

for $k > 2$, and the general existence theorem from Chapter 2 does not guarantee a convex extension for arbitrary functions prescribed on B_n^k .

However, when $k = 2$ (the Boolean hypercube case), the situation changes dramatically. Let $G = \{0, 1\}$ (without loss of generality, by affine transformation). Then $B_n^2 = \{0, 1\}^n$ is precisely the set of all vertices of the unit hypercube $\{0, 1\}^n$. In this special case we do have

$$B_n^2 = \text{vert conv } B_n^2,$$

and the general existence theorem applies. Moreover, the Boolean case admits particularly simple and explicit convex extensions, even for multilinear polynomials.

Let the original function be given on B_n^2 in the multilinear form

$$\varphi(x) = \sum_{1 \leq i_1 \dots < i_{m_p} \leq n} c_{i_1 \dots i_{m_p}} x_{i_1} \dots x_{i_{m_p}}$$

where $1 \leq m_p \leq n$ and each coefficient $c_{i_1 \dots i_{m_p}} \in \mathbb{R}$.

Our goal is to construct an explicit convex function $\tilde{\varphi}(x)$ on $\mathbb{R}^n$ such that $\varphi(x) = \tilde{\varphi}(x)$ for all $x \in B_n^2$.

The construction proceeds by treating each monomial separately according to the sign of its coefficient.

For a monomial with coefficient $\alpha \geq 0$ we have the identity

$$\alpha x_{i_1} \dots x_{i_{m_p}} = \alpha g_{i_1 \dots i_{m_p}}^+(x),$$

where the auxiliary function is defined by

$$g_{i_1 \dots i_{m_p}}^+(x) = \left(\max \left\{ 0, \sum_{j=1}^{m_p} x_{i_j} - m_p + 1 \right\} \right)^2. \quad (4.35)$$

We verify the identity on $\{0, 1\}^n$ by checking the two cases.

If $x_{i_j} = 1$ for all j , then

$$\sum_{j=1}^{m_p} x_{i_j} = m_p,$$

the argument of $\max$ in (4.35) equals 1, and $g_{i_1 \dots i_{m_p}}^+(x) = 1$.

If $x_{i_j} = 0$ for some j , then

$$\sum_{j=1}^{m_p} x_{i_j} \leq m_p - 1,$$

the argument of $\max$ is non-positive, and $g_{i_1 \dots i_{m_p}}^+(x) = 0$.

The function $g_{i_1 \dots i_{m_p}}^+(x)$ is the square of a nonnegative linear form and is therefore convex and differentiable on the whole $\mathbb{R}^n$.

For a monomial with negative coefficient α we have the identity

$$\alpha x_{i_1} \dots x_{i_{m_p}} = -\alpha g_{i_1 \dots i_{m_p}}^-(x) + \alpha, \quad (4.36)$$

where

$$g_{i_1 \dots i_{m_p}}^-(x) = (\max\{0, 1 - x_{i_1}, \dots, 1 - x_{i_{m_p}}\})^2. \quad (4.37)$$

On $\{0, 1\}^n$ both sides of (4.36) coincide. If $x_{i_j} = 1$ for all j , then each term $1 - x_{i_j} = 0$, so $g_{i_1 \dots i_{m_p}}^-(x) = 0$, and $\alpha x_{i_1} \dots x_{i_{m_p}} = \alpha$.

If at least one variable $x_{i_j} = 0$ then $\max$ in (4.37) equals 1, so $g_{i_1 \dots i_{m_p}}^-(x) = 1$, and the right-hand side of (4.36) equals 0.

The function $g_{i_1 \dots i_{m_p}}^-(x)$ is also convex and differentiable on $\mathbb{R}^n$.

Combining the two cases, the desired convex extension $\tilde{\varphi}(x)$ of the original polynomial $\varphi(x)$ is given by

$$\tilde{\varphi}(x) = \sum_{1 \leq i_1 < \dots < i_{m_p} \leq n} c_{i_1 \dots i_{m_p}} g_{i_1 \dots i_{m_p}}(x)$$

where for each multi-index we set

$$g_{i_1 \dots i_{m_p}}(x) = \begin{cases} g_{i_1 \dots i_{m_p}}^+(x) & \text{if } c_{i_1 \dots i_{m_p}} \geq 0; \\ 1 - g_{i_1 \dots i_{m_p}}^-(x) & \text{if } c_{i_1 \dots i_{m_p}} < 0. \end{cases}$$

Each term is convex, hence $\tilde{\varphi}(x)$ is convex on the whole $\mathbb{R}^n$ and coincides with $\varphi(x)$ on B_n^2 by construction.

To obtain a strongly convex extension we add a curvature term that vanishes on the configuration. Define

$$h(x) = \sum_{i=1}^n (x_i^2 - x_i).$$

On B_n^2 we have

$$x_i^2 = x_i,$$

so

$$h(x) = 0.$$

The function $h(x)$ is strongly convex with parameter 2. Therefore, for any $\rho > 0$ the function

$$\Psi(x) = \varphi(x) + \rho h(x)$$

is a strongly convex extension of $\varphi(x)$ to $\mathbb{R}^n$ with strong convexity parameter ρ .

The same technique extends immediately to the general unbounded case B_k^n with arbitrary G by applying the same replacement rules coordinatewise. These explicit formulas close the constructive part of the theory for the unbounded regime and provide the simplest and most transparent convex extensions in the entire hierarchy.

The two special classes examined above, namely the partial permutation configuration E_n^{n+1} and the unbounded configuration B_k^n (especially the Boolean case $k = 2$), are the only regimes within the partial and unbounded classes for which the vertex-located property holds in general.

In all other partial multipermutation configurations E_n^k with nontrivial multiplicity bounds the set is generally not vertex-located, and convex extensions may fail to exist for certain functions. The constructive schemes presented here therefore complete the picture: whenever the configuration is vertex-located, we possess fully explicit analytic extensions (projection for E_n^{n+1} , product-type and quadratic-invariant for B_k^n , with particularly elegant formulas for the Boolean hypercube) that can be used for extremal estimates, stability analysis, and numerical optimization. Together, the results of §§4.1–4.7 form a complete and coherent picture of convex extension and extremal analysis across the full hierarchy of permutation-type configurations.

The constructive extension schemes developed in this chapter cover the full hierarchy of permutation-type configurations. For full multipermutation configurations, polynomial, quadratic, and strongly convex extensions were built explicitly via coordinate-wise operations and curvature augmentation. For partial permutation configurations, the projection technique reduces extension construction to a lower-dimensional multipermutation problem, while exchange arguments and reductions to classical assignment problems govern the extremal analysis. For unbounded and Boolean configurations, coordinate independence yields closed-form extensions of complete analytic transparency, providing the simplest and most computationally explicit instances of the theory across the entire hierarchy.

Extremal ordering principles were established systematically for linear, separable, distance-type, and general convex objectives across all configuration classes. For full configurations, minimization reduces to ordering rules governed by coefficient rearrangements. For partial configurations without repetitions, analogous exchange arguments were established, and quadratic objectives were shown to reduce to classical assignment problems. For unbounded configurations, complete coordinate independence yields closed-form solutions, providing the most computationally transparent extremal rules in the entire hierarchy.

Lower bounds and sufficient optimality conditions for convex extensions were derived systematically across all configuration classes, revealing how convex continuation interacts with discrete admissibility constraints at each level of the hierarchy. These estimates and conditions constitute a complete duality between lower estimation and optimality verification, making the framework self-contained and directly applicable to the optimization methods constructed in Chapter 5.

Chapter 5

Optimization Methods on Combinatorial Point Configurations

The extremal properties and convex extension schemes developed in Chapter 4 transform discrete optimization problems on combinatorial point configurations into equivalent continuous convex problems on polyhedral domains. The present chapter exploits this transformation systematically, constructing a family of optimization methods that operate on the convex carriers of the principal configuration classes and recover global minimizers of the original combinatorial problems.

The chapter opens with a precise formulation of the Euclidean combinatorial optimization problem and its equivalent convex reformulation via the extension theory of Chapter 4 (§5.1). The vertex-located property, established for all principal configuration classes in Chapter 3, guarantees the existence of convex extensions for arbitrary objective and constraint functions, making the reformulation universally applicable within the framework of this monograph.

Two relaxation strategies are developed for polyhedral spherical configurations. The polyhedral relaxation replaces the direct combinatorial constraint by its convex hull and solves the resulting convex program over the carrier polyhedron (§5.2). The spherical relaxation exploits the spherical carrier geometry, reducing the problem to a sequence of trust-region subproblems solvable via secular equations (§5.3). Both relaxations provide lower bounds for the original problem and serve as the bounding engine within branch-and-bound schemes. A unified dimension reduction framework is then developed for multipermutation configurations, reducing the search space through affine projections and embedding the relaxation into a complete branch-and-bound algorithm (§5.4). The branch-and-bound scheme is extended to unbounded partial multipermutation configurations in §5.5, where coordinate independence allows direct application of the multipermutation machinery to each coordinate stratum. The chapter concludes with a penalty-function method for polyhedral spherical configurations, in which the direct constraint is incorporated into the objective via a penalty term and the resulting unconstrained problem is solved iteratively (§5.6).

Together, the methods of this chapter form a complete optimization toolkit for combinatorial point configurations. Each method exploits a specific geometric

feature of the configuration class, demonstrating that the geometric and analytic framework developed throughout the monograph translates directly into practically implementable algorithms.

§5.1. Problem Statement and Nonlinear Programming Models

Let a functional $F : C \rightarrow \mathbb{R}$ be defined on a combinatorial set C . Here the term functional is used deliberately to distinguish F from functions defined on $\mathbb{R}^n$: F maps abstract combinatorial objects to real numbers, whereas the function f introduced below maps vectors in $\mathbb{R}^n$ to real numbers. The combinatorial optimization problem is to find

$$\pi^* = \arg \min F(\pi), \quad \pi \in C', \quad (5.1)$$

where $C' \subseteq C$ is the feasible set.

Consider a bijective mapping

$$\varphi : C \rightarrow E \subset \mathbb{R}^n$$

and a function $f : E \rightarrow \mathbb{R}$ satisfying

$$f(\varphi(\pi)) = F(\pi) \text{ for all } \pi \in C.$$

Then problem (5.1) admits the equivalent formulation: find

$$x^* = \arg \min f(x), \quad x \in E', \quad (5.2)$$

where $E' = \varphi(C')$ and $E = \varphi(C)$ is the combinatorial point configuration (CPC), with $|E| = |C|$.

Problem (5.2) is called a Euclidean combinatorial optimization problem (ECOP). Thus ECOPs constitute a special class of discrete optimization problems whose feasible domain is a finite point configuration in $\mathbb{R}^n$.

The mapping φ is not unique: infinitely many bijections $C \rightarrow \mathbb{R}^n$ satisfy the identity

$$f(\varphi(\pi)) = F(\pi)$$

when f is formally defined via the inverse as

$$f(x) = F(\varphi^{-1}(x)).$$

Such a formal construction, however, is insufficient for continuous optimization: what is required is a mapping φ for which the induced function f admits an

explicit analytical representation on $\mathbb{R}^n$ that extends continuously to the convex hull $\text{conv } E$.

The existence of such a representation is problem-dependent and constitutes a key modelling step. At the same time, the image $E = \varphi(C)$ must be vertex-located, since this property is essential for the convex extension theory developed in Chapter 4 to apply. The selection of an appropriate φ thus simultaneously determines the geometry of E and the analytical tractability of f .

For the principal configuration classes studied in this monograph – multipermutation and Boolean configurations – both conditions are verified: the vertex-located property is established in Chapter 3, while the analytical form of f is specified explicitly for each problem class in the subsequent sections. Moreover, even when a combinatorial point configuration is not vertex-located, it always admits a decomposition into vertex-located subsets, as established in Chapter 1 for arbitrary finite point configurations and in Chapter 3 for partial multipermutation point configurations.

In addition to the direct constraint $x \in E$, practical problems may involve functional constraints. Note that the functions f and g_i are defined on E ; their extensions to a convex set $X \supseteq P$, where $P = \text{conv } E$, which in general differ from the original functions outside E but coincide with them on E , are introduced in Theorem 5.1 below. When the analytical expression of f or g_i extends directly to X without modification, the extension coincides with the original function on X ; otherwise a convex extension must be constructed explicitly by the methods of Chapter 4.

The general ECOP is therefore formulated as

$$\min f(x), \quad x \in X, \quad (5.3)$$

where the feasible set X is defined by the system of constraints:

$$x \in E, \quad (5.4)$$

$$g_i(x) \leq 0, \quad i \in J_k, \quad (5.5)$$

$$g_i(x) = 0, \quad i \in J_m \setminus J_k, \quad (5.6)$$

where $0 \leq k \leq m$, and the functions $f(x)$ and $g_i(x)$ for $i \in J_m$ are defined on E .

Throughout this monograph J_n denotes the standard index set $\{1, 2, \dots, n\}$. Constraint (5.4) is called the direct constraint, while (5.5) and (5.6) are called functional constraints.

A special case of model (5.3)–(5.6) arises when the CPC $E \subset \mathbb{R}^n$ is vertex-located, that is, when it coincides with the vertex set of its convex hull:

$$E = \text{vert}(\text{conv } E). \quad (5.7)$$

As shown in Chapter 3, all principal classes of CPCs considered in this monograph are vertex-located. These include permutation and multipermutation configurations E_n and E_{nk} , partial permutation configurations E_n^{n+1} , and unbounded

partial multipermutation configurations B_n^2 . The vertex-located property is crucial because it guarantees the existence of convex extensions for arbitrary functions defined on E , as established in Chapter 4. This allows the original problem (5.3)–(5.6) to be reformulated equivalently with convex objective and constraint functions while preserving all global minimizers on E .

The following theorem makes this reformulation precise. Throughout the remainder of the chapter we will work exclusively with the equivalent convex formulation it provides.

Theorem 5.1. (*Equivalent convex formulation*).

Let E be a vertex-located CPC and let problem (5.3)–(5.6) be given. Then this problem is equivalent to the following optimization problem:

$$\min \tilde{f}(x), \quad x \in \tilde{X}, \quad (5.8)$$

where the feasible set $\tilde{X}$ differs from X only in that the original constraint functions are replaced by their convex extensions: it is defined by

$$x \in E, \quad (5.9)$$

$$\tilde{g}_i(x) \leq 0, \quad i \in J_r, \quad (5.10)$$

where $r = 2m - k$ is the total number of inequality constraints after converting each equality constraint into a pair of inequalities, and $\tilde{f}$ and $\tilde{g}_i$ for $i \in J_r$ are convex extensions of f and g_i from E onto any convex set $X \supseteq P = \text{conv } E$.

Proof. Each equality constraint $g_i(x) = 0$ with $i \in J_m \setminus J_k$ is equivalent to the pair of inequalities $g_i(x) \leq 0$ and $-g_i(x) \leq 0$. By the fundamental theorem on convex extensions for vertex-located sets (Chapter 4), there exist convex functions $\tilde{g}_i$ defined on X such that those corresponding to original inequality constraints coincide with g_i on E , while those generated from each equality constraint coincide with g_i and $-g_i$ respectively.

Since there are $m - k$ equality constraints and each generates two inequalities, the total number of inequality constraints becomes $k + 2(m - k) = 2m - k$.

The same argument applies to f and to all original inequality constraints. Consequently, for any $x \in E$ the original feasibility conditions hold if and only if the corresponding convex inequalities $\tilde{g}_i(x) \leq 0$ hold, and since f and $\tilde{f}$ coincide on E , the two problems share the same global minimizers within E . ■

To avoid cluttering the notation with tildes over f and g_p in the subsequent sections we use formulation (5.3)–(5.5) without equality constraints, assuming that all constraint functions are convex. This assumption entails no loss of generality in view of Theorem 5.1, since any equality constraint can always be represented as a pair of convex inequalities.

The convex formulation provided by Theorem 5.1 opens the way to continuous relaxations. For the CPC classes considered here, which as shown in Chapter 1 and Chapter 3 all admit the polyhedral-spherical representation

$$E = P \cap S_r(\tau), \quad (5.11)$$

two fundamentally different relaxations are possible, each obtained by retaining one component of intersection (5.11) and dropping the other.

Since $E \subset P$ and $E \subset S_r(\tau)$, any relaxation that expands the feasible set from E to either P or $S_r(\tau)$ yields a problem whose optimal value cannot exceed z^* , because minimizing over a larger set can only decrease or preserve the minimum. This fundamental observation guarantees that both relaxation bounds are valid lower bounds on the discrete optimum z^* .

The first relaxation, developed in detail in §5.2, is obtained by dropping the spherical constraint $x \in S_r(\tau)$ and solving the problem over the polytope P alone. Since all functions are convex by Theorem 5.1, the resulting problem is a convex programme. Its optimal value z_p satisfies $z_p \leq z^*$ and therefore provides a valid lower bound on the discrete optimum z^* . Moreover, if the minimizer of this relaxation lies in E , it is simultaneously the exact global solution of the original problem.

The second relaxation, developed in detail in §5.3, retains the hypersphere $S_r(\tau)$ and drops the polyhedral constraint $x \in P$. In general this problem is multiextremal. However, for quadratic objective functions it reduces to finding the eigenvector corresponding to the smallest eigenvalue of the quadratic form matrix and admits an explicit analytical solution. The resulting minimum z_s satisfies $z_s \leq z^*$ and provides a second valid lower bound on z^* . The detailed analysis of this reduction and the conditions under which the spherical bound is sharp are given in §5.3.

Since neither relaxation dominates the other in general, both are solved and their results are combined. The combined lower bound

$$z_l = \max \{z_p, z_s\} \quad (5.12)$$

is tighter than either bound alone, since both $z_p \leq z^*$ and $z_s \leq z^*$ hold simultaneously, and taking the maximum preserves the lower bound property while improving its quality. The relative tightness of the two bounds depends on the geometry of the specific problem instance, as analysed in detail in §5.3.

The two relaxations alone do not constitute a complete algorithm: when the combined lower bound (5.12) does not yet certify optimality, the search must be continued by decomposing the problem. This is achieved by partitioning E into subsets through families of parallel hyperplanes. For the multipermutation configurations $E_{nk}(G)$, Theorem 3.6 of Chapter 3 establishes that for any index subset $\omega \subset J_n$ with $|\omega| = s$, the configuration decomposes into the disjoint union

$$E_{nk}(G) = \bigcup_{t \in \Sigma_{G,s}} E^{t,\omega},$$

$$E^{t,\omega} = E_{n_k}(G) \cap \left\{ x : \sum_{i \in \omega} x_i = t \right\}.$$

where each stratum $E^{t,\omega}$ is itself a vertex-located CPC of strictly smaller cardinality.

An analogous stratification holds for Boolean and partial multipermutation configurations, as established in Chapter 3. Each element of the partition defines a subproblem of exactly the same structure as the original, but on a strictly smaller configuration, to which the two relaxations are applied recursively. A subproblem is eliminated from further consideration whenever its local lower bound exceeds the best upper bound known at that point, which is maintained as the objective value at the best feasible point found so far.

Each branching hyperplane adds an equality constraint, reducing the dimension of the feasible affine subspace by one. After j steps the subproblem lives in a space of dimension $n - j$. An orthogonal transformation – chosen precisely because it preserves the hypersphere $S_r(\tau)$ and hence the applicability of the spherical relaxation of §5.3 – then converts this affine subproblem into a standard spherical optimization problem in reduced dimension, so that the eigenvalue solution of §5.3 applies without modification. The construction of this orthogonal transformation is detailed in §5.4.

The lower bounds are non-decreasing along every path in the branching tree, since adding equality constraints to the polyhedral relaxation can only increase z_p , and restricting the sphere to a lower-dimensional subsphere can only increase z_s as the feasible set shrinks at each level. Since E is finite, the branching tree has depth at most $n - 1$: after $n - 1$ branching steps each node contains at most one element of E , by the properties of the hyperplane families established in Chapter 3 (Theorem 3.6). The number of nodes at each level is bounded by the number of hyperplanes in the corresponding family, so the total number of nodes is finite and the algorithm terminates in finitely many steps with a globally optimal or ε -optimal solution.

The sufficient optimality conditions of Chapter 4 provide an additional mechanism for early termination that complements the branch-and-bound scheme. Specifically, Theorems 4.2 and 4.6 establish that a candidate solution $x^* \in E$ is a global minimizer of f on E if the components of the gradient of the convex extension φ at x^* are ordered oppositely to the coordinates of x^* – the so-called gradient ordering condition. Precisely, there must exist a permutation $(\pi_1, \dots, \pi_n)$ of $\{1, \dots, n\}$ such that simultaneously

$$\frac{\partial \varphi(x^*)}{\partial x_{\pi_1}} \geq \dots \geq \frac{\partial \varphi(x^*)}{\partial x_{\pi_n}},$$

$$x_{\pi_1}^* \leq \dots \leq x_{\pi_n}^*.$$

This condition can be verified in $O(n \log n)$ time by a single sorting operation, making it computationally inexpensive. Moreover, the gradient-based lower bound estimates of Theorems 4.1, 4.8, and 4.9 provide, at every node of the branching tree, explicit lower bounds on the minimum expressible in terms of φ and its gradient at any current point – again computable by sorting or by solving a bounded assignment problem. This combination of relaxations (5.11)–(5.12), branching, dimension reduction, and gradient-based optimality certificates from Chapter 4 forms the complete algorithmic backbone developed in §§5.2–§5.5.

§5.2. Polyhedral Relaxation for Polyhedral-Spherical Configurations

The present section develops the polyhedral relaxation approach for optimization on vertex-located combinatorial point configurations, and then specializes it to the two principal classes studied in this monograph: multipermutation configurations $E_{nk}(G)$ and Boolean configurations B_k^n . The general framework is presented first, since its key structural property, namely the reduction of every auxiliary subproblem to an explicit extremal computation on E , holds for any vertex-located CPC and underlies all subsequent specializations. The section concludes with a discussion of the lower and upper bounds arising from the convex extension theory of Chapter 4.

Let E be a vertex-located CPC with $P = \text{conv } E$ and polyhedral-spherical representation (5.11). In accordance with the convention of §5.1, we assume without loss of generality (by Theorem 5.1) that the objective function f and all constraint functions g_i are convex on any convex set $X \supseteq P$. The polyhedral relaxation is obtained by dropping the spherical constraint from the representation of E and optimizing over P alone:

$$\min f(x), \quad x \in P, \quad g_i(x) \leq 0, \quad i \in J_r. \quad (5.13)$$

Since f and all g_i are convex on P , problem (5.13) is a convex programming problem, and its optimal value z_p satisfies

$$z_p \leq z^*$$

because $E \subset P$.

Two iterative methods are applicable to (5.13). In the case without functional constraints, that is, the problem $\min f(x), x \in P$, both methods require solving an auxiliary subproblem at every iteration.

The conditional gradient method (Frank–Wolfe method) requires at each iteration the solution of the linear subproblem

$$\arg \min_{y \in P} \langle f'(x^{(l)}), y \rangle. \quad (5.14)$$

The gradient projection method requires the nearest-point computation

$$Pr_E c^{(l)} = \arg \min_{y \in E} \|y - c^{(l)}\|^2, \quad (5.15)$$

where $c^{(l)} = x^{(l)} - \alpha_l f'(x^{(l)})$ is the gradient step point.

The key structural observation that makes both methods efficient on vertex-located configurations is the following. Since $E = \text{vert } P$, any linear function attains its minimum over the convex hull P at a vertex, hence at a point of E . This gives the identity

$$\arg \min_{y \in E} \langle c, y \rangle = \arg \min_{y \in P} \langle c, y \rangle, \quad (5.16)$$

established as the Corollary to Lemma 4.1 in Chapter 4.

Identity (5.16) means that for the conditional gradient method, solving the linear subproblem (5.14) over the polytope P automatically yields a vertex of P , that is, an element of E . No rounding or additional projection step is needed. This property is specific to linear objectives and is a direct consequence of the vertex-located property of E . It does not extend to the nearest-point subproblem (5.15) of the gradient projection method: in general,

$$\arg \min_{y \in E} \|y - c\|^2 \neq Pr_P c,$$

since a quadratic function need not attain its minimum over a polytope at a vertex.

Therefore the gradient projection method requires an explicit nearest-point computation on E rather than on P . The efficiency of this computation depends entirely on the specific geometry of the configuration E , and for the two principal classes studied here it reduces to sorting or coordinatewise selection by virtue of the distance function lemmas of Chapter 4.

The sufficient optimality condition at a candidate point $x^* \in E$ takes the following unified form for both methods. By the general theory of Chapter 4, a point x^* is a global minimizer of f on E if x^* itself solves the linear subproblem (5.16) with $c = \varphi'(x^*)$, that is, if x^* already attains the minimum of the linear functional $\langle \varphi'(x^*), y \rangle$ over E . The specific form of this condition for each configuration class is given by the corresponding theorems of Chapter 4 recalled below.

We now specialize the general framework to multipermutation configurations $E_{nk}(G)$. Recall from Chapter 3 that this configuration is generated by a numerical multiset $G = \{g_1^{n_1}, \dots, g_k^{n_k}\}$, with underlying set $S(G) = \{g_1 < \dots < g_k\}$, specification

$[G] = (n_1, \dots, n_k)$, and $n_1 + \dots + n_k = n$. Listing the elements of G with their multiplicities in nondecreasing order gives the ordered representation

$$A = (a_1 \leq \dots \leq a_n), \quad (5.17)$$

and every point $x \in E_{nk}(G)$ is a permutation of A .

We set

$$S_1 = \sum_{i=1}^n a_i, \quad S_2 = \sum_{i=1}^n a_i^2.$$

By Theorem 3.5 of Chapter 3, the configuration admits the polyhedral-spherical representation

$$E = P \cap S_r(\tau), \quad (5.18)$$

where $\Pi_{nk}(G) = \text{conv } E_{nk}(G)$ is the generalized permutohedron, and $S_r(\tau)$ is the circumscribed hypersphere from the family (3.4) with centre τ and radius r , given by Theorem 3.3,

$$\tau = \frac{S_1}{n} r^2 = S_2 - \frac{S_1^2}{n}.$$

Thus every point of $E_{nk}(G)$ lies simultaneously on the boundary of the convex polytope $\Pi_{nk}(G)$ and on the hypersphere $S_r(\tau)$. This dual membership is the geometric foundation for both relaxation approaches developed in this chapter. The centroid τ is the centre of the minimum circumsphere and belongs to the interior of $\Pi_{nk}(G)$, providing a natural and always-available initial point for iterative methods.

The H-representation of $\Pi_{nk}(G)$ is given by Theorem 3.9 of Chapter 3. The generalized permutohedron is the set of all $x \in \mathbb{R}^n$ satisfying the affine equation (3.2) together with the inequalities (3.7). Although the system may contain up to $2^n - 1$ constraints, Theorem 3.9 identifies the irredundant representation. In practice, this reduction substantially decreases the number of active constraints and makes the relaxation problem computationally tractable even for moderate values of n . The polyhedral relaxation of the optimization problem on $E_{nk}(G)$ takes the form (5.13) with $P = \Pi_{nk}(G)$.

It is worth emphasising the conceptual simplicity of this relaxation: we have replaced a discrete optimization problem over a finite set of $n! / (n_1! \cdot \dots \cdot n_k!)$ points by a continuous convex programming problem over a polytope. The price we pay is that the relaxation solution may not be feasible for the original problem; the gain is that convex programming problems can be solved efficiently with global optimality guarantees.

For the conditional gradient method, the linear subproblem (5.14) over $\Pi_{nk}(G)$ is solved by Lemma 4.1 of Chapter 4 (with identity (5.16) ensuring the result is an

element of $E_{nk}(G)$: the minimizer is found by sorting the gradient components in non-increasing order according to conditions (4.7)–(4.8) and assigning a_1 to the position with the largest gradient component, a_2 to the second largest, and so on. This operation requires only $O(n \log n)$ arithmetic operations, the cost of sorting, and no linear programming solver is needed.

For the gradient projection method, the nearest-point subproblem (5.15) on $E_{nk}(G)$ is solved by Lemma 4.3 of Chapter 4, whose Remark 4.3 identifies it precisely as the Euclidean projection of c onto the vertex set of $\Pi_{nk}(G)$: the minimizer is found by assigning the largest elements of A to the largest components of c , as expressed by conditions (4.11)–(4.12). This operation likewise costs $O(n \log n)$. Thus both auxiliary subproblems for $E_{nk}(G)$ reduce to a single sorting operation, making each iteration of both methods computationally cheap regardless of the size of the H -representation of $\Pi_{nk}(G)$.

We now describe the two methods in detail for problem

$$\min f(x), \quad x \in \Pi_{nk}(G). \quad (5.19)$$

The conditional gradient method generates a sequence in $\Pi_{nk}(G)$ converging to the global minimum off over $\Pi_{nk}(G)$ at a rate $O(1/l)$ in function value. Starting from $x^{(0)} \in \Pi_{nk}(G)$, for example the centroid τ , at iteration l the method solves the linear subproblem (5.14) by sorting according to (4.7)–(4.8), obtaining $y^{(l)} \in E_{nk}(G)$, and forms the new iterate

$$x^{(l+1)} = (1 - \alpha_l) x^{(l)} + \alpha_l y^{(l)}, \quad (5.20)$$

where the step size $\alpha_l \in [0,1]$ is found by line search

$$\alpha_l = \arg \min_{\alpha \in [0,1]} f((1-\alpha)x^{(l)} + \alpha y^{(l)}). \quad (5.21)$$

For the quadratic objective function

$$(x) = x^T C x + x^T b,$$

the line search (5.21) admits an explicit solution.

Setting $h^{(l)} = y^{(l)} - x^{(l)}$, the objective along the line becomes

$$f(\alpha) = \alpha^2 (h^{(l)})^T C h^{(l)} + \alpha (2(x^{(l)})^T C h^{(l)} + (h^{(l)})^T b) + f(x^{(l)}).$$

Denoting

$$\gamma = -(2(x^{(l)})^T C h^{(l)} + (h^{(l)})^T b) / (2(h^{(l)})^T C h^{(l)}),$$

the optimal step size equals 0 if $\gamma \notin (0,1)$ and the directional derivative is positive, 1 if $\gamma \notin (0,1)$ and the directional derivative is negative, and γ if $\gamma \in (0,1)$ and $(h^{(l)})^T C h^{(l)} > 0$.

For a general convex differentiable f , the line search (5.21) is carried out by any standard one-dimensional method, such as the golden section method or bisection.

tion, since f is convex along every line segment. The iteration is repeated until the duality gap

$$\delta^{(l)} = f(x^{(l)}) - \langle f'(x^{(l)}), x^{(l)} - y^{(l)} \rangle \quad (5.22)$$

falls below a prescribed tolerance $\varepsilon > 0$.

The duality gap (5.22) provides a computable upper bound on the distance from the current function value to the true minimum over $\Pi_{nk}(G)$ and is therefore a reliable stopping criterion.

The gradient projection method maintains iterates directly in $E_{nk}(G)$ by projecting each gradient step onto the configuration. Starting from $x^{(0)} \in E_{nk}(G)$, at iteration l the method computes the gradient step point

$$c^{(l)} = x^{(l)} - \alpha_l f'(x^{(l)})$$

and obtains the new iterate by solving the nearest-point subproblem (5.15):

$$x^{(l+1)} = Pr_{E_{nk}(G)} c^{(l)} = \arg \min_{y \in E_{nk}(G)} \|y - c^{(l)}\|^2. \quad (5.23)$$

By Lemma 4.3 and conditions (4.11)–(4.12), the projection (5.23) costs $O(n \log n)$. The step size α_l may be chosen as a constant satisfying $0 < \alpha_l \leq 1/L$ where L is the Lipschitz constant of f' , or by line search along the projection arc. There is a direct connection between the gradient projection step and the lower bound estimates of Chapter 4: when f possesses a strongly convex extension φ with parameter $\rho > 0$, the centre $c^{(l)}$ with $\alpha_l = 1/(2\rho)$ coincides exactly with the shifted centre

$$c = x - \frac{\varphi'(x)}{2\rho}$$

of Theorem 4.8, whose lower bound is (4.25).

Thus the nearest-point computation (5.23) simultaneously advances the iterate and contributes to the lower bound estimate. Moreover, Theorem 4.10 gives a sufficient condition for global optimality native to the gradient projection method: a point $x^* \in E_{nk}(G)$ is a global minimizer if it satisfies condition (4.28), verified at each iteration at no additional cost.

When functional constraints in (5.13) are present, both methods are extended by incorporating the constraints into the objective via a penalty term. For a penalty parameter $\alpha > 0$, the penalized problem is

$$\min_{x \in \Pi_{nk}(G)} (f(x) + \psi(x, \alpha)), \quad (5.24)$$

$$\psi(x, \alpha) = \alpha \sum_{i=1}^r (\max\{0, g_i(x)\})^2.$$

The use of the squared violation ensures that $\psi(\cdot, \alpha)$ is both convex and continuously differentiable on $\Pi_{nk}(G)$, preserving the convexity structure required by both methods. As $\alpha \rightarrow \infty$, by the classical theory of exterior penalty functions, the minimizers of (5.24) converge to the solution of the constrained problem (5.13). In practice, one solves a sequence of problems (5.24) with increasing α , using each solution as the starting point for the next. For any fixed α , problem (5.24) is a convex programming problem over $\Pi_{nk}(G)$, and both methods apply directly with the gradient of the penalized objective

$$f'(x, \alpha) + \psi'(x, \alpha) = f'(x) + 2\alpha \sum_{i=1}^r \frac{\partial g_i(x)}{\partial x_i} \max\{0, g_i(x)\}. \quad (5.25)$$

The auxiliary subproblems of both methods remain linear minimization or nearest-point computations on $\Pi_{nk}(G)$ or $E_{nk}(G)$ respectively, solved by sorting via Lemmas 4.1 and 4.3. Both methods therefore inherit their computational efficiency from the unconstrained case, adding only the evaluation of ψ and its gradient (5.25) at each iteration.

The polyhedral relaxation provides the lower bound $z_p \leq z^*$. Sharper bounds follow from the convex extension theory of Chapter 4.

By Theorem 4.1, if $\varphi: X \rightarrow \mathbb{R}$ is a convex differentiable extension of f to $X \supseteq E_{nk}(G)$, the lower bound estimate (4.13) holds at every $x \in X$. The minimum of $\langle \varphi'(x), y \rangle$ over $E_{nk}(G)$ on the right-hand side of (4.13) is evaluated by sorting according to (4.14) via Lemma 4.1 in $O(n \log n)$.

Substituting the current iterate $x^{(l)}$ yields the computable lower bound

$$z_l = \varphi(x^{(l)}) + \sum_{i=1}^r \frac{\partial \varphi(x^{(l)})}{\partial x_{k_i}} (a_i - x_{k_i}^{(l)}) \leq z^*. \quad (5.26)$$

For the upper bound, the value of f at the nearest element of $E_{nk}(G)$ to the current iterate provides $z_u \geq z^*$. By Lemma 4.3 and conditions (4.11)–(4.12), this nearest element is found by sorting at cost $O(n \log n)$. When the gradient projection method is used, the nearest element is computed automatically at each iteration (5.23), so z_u is obtained as a by-product at no additional cost. Together,

$$z_l \leq z^* \leq z_u \quad (5.27)$$

and the gap $z_u - z_l$ measures the current approximation accuracy. Beyond bounding, Theorem 4.2 provides a constructive sufficient condition for global optimality: a point $x^* \in E_{nk}(G)$ is a global minimizer if conditions (4.15)–(4.16) hold simultaneously, that is, the gradient of φ at x^* is ordered oppositely to the coordinates of x^* . This means x^* already attains the minimum of $\langle \varphi'(x^*), y \rangle$ over $E_{nk}(G)$, so convexity of φ implies global optimality. Conditions (4.15)–(4.16) are checked by a single comparison of orderings whenever the algorithm produces a candidate point $x^* \in E_{nk}(G)$, and if satisfied, the algorithm terminates immediately with x^* certified as the global solution.

Both the conditional gradient method and the gradient projection method apply without modification to Boolean configurations. These configurations arise as Cartesian products

$$B_k^n = G^n$$

of a finite set $G = \{g_1 < \dots < g_k\}$, with each coordinate x_i chosen independently from G . As established in Chapter 3 (§3.7), B_k^n is vertex-located and admits the polyhedral-spherical representation

$$B_k^n = P \bigcap \mathcal{S}_r(\tau),$$

so the general framework of this section applies.

The decisive structural difference from $E_{nk}(G)$ is the complete independence of coordinates in B_k^n : no admissibility constraint links different coordinates, so both auxiliary subproblems decompose completely into n independent one-dimensional problems since the coordinates of B_k^n are chosen independently. This reduces the complexity of all auxiliary computations from $O(n \log n)$ to $O(n)$.

For the conditional gradient method, the linear subproblem (5.14) over B_k^n is solved by Lemma 4.12 according to condition (4.29): for each coordinate $i = 1, \dots, n$ independently,

$$y_i^* = \begin{cases} g_1 & \text{when } \frac{\partial f(x)}{\partial x_i} \geq 0; \\ g_k & \text{when } \frac{\partial f(x)}{\partial x_i} < 0. \end{cases}$$

No sorting across coordinates is required; each coordinate is decided by the sign of the corresponding gradient component alone. Identity (5.16) continues to hold for B_k^n : since

$$B_k^n = \text{vert conv } B_k^n,$$

the linear minimization over B_k^n and over its convex hull give the same result, and the minimizer is automatically a vertex.

For the gradient projection method, the nearest-point subproblem (5.15) on B_k^n is solved by Lemma 4.13 according to condition (4.30): for each coordinate $i = 1, \dots, n$ independently, the nearest element of G to c_i is found by inspection, requiring $O(1)$ per coordinate and $O(n)$ overall. As noted in §4.6 this replaces the assignment problem that arises for $E_{nk}(G)$ (Lemma 4.3), cleanly demonstrating the structural gain from coordinate independence. The bounds (5.26)–(5.27) apply to B_k^n via Theorem 4.11, with the linear minimization on the right-hand side solved coordinatewise by Lemma 4.12 in $O(n)$.

The sufficient optimality condition is Theorem 4.12 via condition (4.30): a point $x^0 \in B_k^n$ is a global minimizer if for every i ,

$$\frac{\partial \phi(x)}{\partial x_i} \begin{cases} \geq 0 & \text{whenever } x_i^0 = g_1; \\ < 0 & \text{whenever } x_i^0 = g_k, \end{cases}$$

checked in $O(n)$.

For the strongly convex case, Theorem 4.15 gives the fixed-point characterization (4.34), verified at each projection step at no extra cost.

Thus, the polyhedral relaxation approach unifies two configuration classes under a single framework: for any vertex-located CPC E , both the conditional gradient method and the gradient projection method reduce their auxiliary subproblems to explicit extremal computations on E itself. The conditional gradient method is particularly natural because identity (5.16) – a direct consequence of the vertex-located property – ensures the linear subproblem automatically yields a vertex without any additional projection.

The gradient projection method complements this with a nearest-point step that simultaneously provides the upper bound and the optimality certificate, and its iterate structure is intrinsically connected to the lower bound estimates of Theorem 4.8 through the shifted centre.

For multipermutation configurations all auxiliary computations cost $O(n \log n)$. For Boolean configurations coordinate independence reduces this to $O(n)$, as expressed concisely by the contrast between Lemma 4.3 and Lemma 4.13. Together, the convex relaxation over P , the lower bound (5.26) from convex extension theory, and the sufficient optimality conditions from Chapter 4 constitute a theoretically complete and computationally efficient framework for global optimization on vertex-located CPCs, which serves as the foundation for the branch-and-bound scheme developed in §5.4.

§5.3. Spherical Relaxation for Polyhedral-Spherical Configurations

The polyhedral relaxation of §5.2 exploits one component of representation (5.11) by optimizing over the polytope P alone. The present section develops the complementary approach: retaining the hypersphere $S_r(\tau)$ and dropping the polyhedral constraint. The resulting spherical relaxation provides lower bounds that are in general incomparable with those of the polyhedral relaxation, and their combination yields bounds of significantly higher quality than either alone.

The spherical relaxation is natural for polyhedral-spherical configurations – those admitting the representation (5.11). This class includes all principal configuration classes studied in this monograph: multipermutation configurations $E_{nk}(G)$ (Theorem 3.5) and the two-level Boolean configurations B_2^n . For $k > 2$, the general Boolean configuration B_k^n is not spherically located (Chapter 3, §3.7), so the spherical relaxation applies to its two-level subcase $B_2^n \subset B_k^n$ only. The general framework is presented first, and then specialized to each class.

Let E be a polyhedral-spherical CPC with representation (5.11). The spherical relaxation of the optimization problem on E is obtained by dropping the polyhedral constraint $x \in P$ and retaining only the spherical constraint $x \in S_r(\tau)$:

$$\min f(x), \quad x \in S_r(\tau). \quad (5.28)$$

Since $E \subset S_r(\tau)$ by the polyhedral-spherical representation, the feasible set of (5.28) contains E , and therefore

$$z_S = \min_{x \in S_r(\tau)} f(x) \leq z^*.$$

Thus z_S is a valid lower bound on z^* . Similarly, the polyhedral relaxation from §5.2 provides the lower bound $z_p \leq z^*$. Since both bounds are valid simultaneously, their maximum

$$z_l = \max\{z_p, z_S\} \quad (5.29)$$

is also a valid lower bound, and is at least as sharp as either bound individually. The combined lower bound (5.29) is one of the central tools of the branch-and-bound method developed in §5.4.

It is natural to ask when z_S is sharper than z_p and vice versa. The answer depends on the geometry of the problem, specifically on the location of the unconstrained minimizer

$$\hat{x} = \arg \min f(x)$$

relative to the polytope P and the hypersphere $S_r(\tau)$. We formalize this observation in the following theorem.

Theorem 5.2. (*Comparison of polyhedral and spherical lower bounds*).

Let f be a convex differentiable function on $\mathbb{R}^n$, and let

$$\hat{x} = \arg \min f(x)$$

be its unconstrained minimizer.

(i) If $\hat{x} \in \text{int } P$, then

$$z_p = \min_{x \in P} f(x) = f(\hat{x}),$$

so the polyhedral relaxation yields only the unconstrained minimum as a lower bound. Since all points of E lie on the boundary of P and on the sphere $S_r(\tau)$, the

spherical relaxation is likely to provide a sharper bound: the minimum of f over $S_r(\tau)$ is constrained to the sphere passing through E and may be significantly larger than $f(\hat{x})$, giving $z_s \geq z_p$ in this case, so the spherical bound is sharper.

(ii) If $\hat{x} \notin P$, then the unconstrained minimizer lies outside the polytope, and the polyhedral relaxation is likely to be more informative, since the minimum of f over P is attained on the boundary and reflects the geometry of P more directly.

(iii) If $\hat{x} \in \text{int } S_r(\tau)$, meaning the unconstrained minimizer lies strictly inside the hypersphere, then the spherical minimum z_s is attained on $S_r(\tau)$ and is strictly greater than $f(\hat{x})$, providing a non-trivial lower bound. In this case $z_s \geq z_p$ whenever P extends further from $\hat{x}$ than $S_r(\tau)$ does.

(iv) In all cases, the combined bound

$$z_l = \max\{z_p, z_s\}$$

is valid and is never worse than either individual bound.

Proof. Since f is convex and $\hat{x}$ is its unconstrained minimizer, $f(x) \geq f(\hat{x})$ for all $x \in \mathbb{R}^n$, with equality only at $\hat{x}$.

For any feasible set D containing E , we have

$$\min_{x \in D} f(x) \leq z^*$$

so any such minimum provides a lower bound.

In case (i), since $\hat{x} \in \text{int } P$, the minimum of f over P is attained at $\hat{x}$, giving $z_p = f(\hat{x})$. This is the weakest possible lower bound, since $f(\hat{x}) \leq f(x)$ for all x . The spherical relaxation optimizes over $S_r(\tau)$, which passes through all points of E and avoids the interior of P : since f is convex and $\hat{x} \in \text{int } P$, the values of f on $S_r(\tau)$ are strictly greater than $f(\hat{x})$, giving $z_s \geq z_p = f(\hat{x})$. Therefore the spherical bound is strictly sharper in this case.

In case (ii), since $\hat{x} \notin P$ and f is convex, the minimum of f over P is attained on the boundary of P in the direction from $\hat{x}$. The polyhedral structure directly captures the geometry of the feasible set $E = \text{vert } P$, making z_p more directly relevant.

Cases (iii) and (iv) follow by analogous convexity arguments applied to the sphere $S_r(\tau)$. The bound z_l is valid since both z_p and z_s are valid lower bounds, and the maximum of two lower bounds is again a lower bound. ■

The theorem shows that neither relaxation uniformly dominates the other: their relative quality depends on the geometry of the specific problem instance, making the combined bound particularly valuable in practice. In the general case of an arbitrary convex differentiable objective, problem (5.28) is non-convex on a smooth manifold and may be multiextremal. The situation changes fundamentally for quadratic objectives, where problem (5.28) reduces to an eigenvalue problem admitting a complete explicit solution, developed in detail below for both $E_{nk}(G)$ and for B_2^n . For general smooth f , the explicit quadratic solution can still be exploited through local quadratic approximations, as described at the end of this section.

We now specialize to multipermutation configurations $E_{nk}(G)$. Recall from §5.2 that the polyhedral-spherical representation is

$$E_{nk}(G) = \Pi_{nk}(G) \cap S_r(\tau),$$

where the circumscribed hypersphere $S_r(\tau)$ has radius

$$r = \sqrt{S_2 - S_1^2 / n},$$

and centre $\tau = (\tau_1, \dots, \tau_n)$ where

$$\tau_1 = \dots = \tau_n = S_1 / n.$$

The spherical relaxation of the original optimization problem on $E_{nk}(G)$ is obtained by dropping the polyhedral constraint $x \in \Pi_{nk}(G)$ from representation (5.18) and retaining only the spherical constraint $x \in S_r(\tau)$, giving problem (5.28) with this specific $S_r(\tau)$.

We begin with the general case of an arbitrary convex differentiable objective function f , establish the basic properties of the relaxation, and then turn to the quadratic case where the relaxation admits a complete explicit solution. The section concludes with a discussion of the case with additional functional constraints and a remark on the use of quadratic approximations for general objective functions.

Before turning to the explicit solution of the spherical relaxation, we establish an auxiliary result that plays a key role in the branch-and-bound scheme of §5.4 and in the geometric analysis of the spherical relaxation. Specifically, we need to identify the face of the generalized permutohedron $\Pi_{nk}(G)$ that is closest to the centre τ in a given direction $\xi \in S_r(\tau)$.

Recall that τ is the center of the circumscribed hypersphere $S_r(\tau)$, and that $\tau \in \text{int } \Pi_{nk}(G)$ since τ is the centroid of $E_{nk}(G)$. For a given point $\xi \in S_r(\tau)$, the segment $[\tau, \xi]$ starts at the centre of $\Pi_{nk}(G)$ and ends on its circumsphere. Along this segment, the first face of $\Pi_{nk}(G)$ encountered as we move from τ toward ξ is called the nearest face of $\Pi_{nk}(G)$ in the direction τ_ξ .

Definition 5.1. The nearest face of $\Pi_{nk}(G)$ in the direction τ_ξ where $\xi \in S_r(\tau)$, is the face of $\Pi_{nk}(G)$ whose supporting hyperplane has non-empty intersection with the open segment (τ, ξ) and whose intersection point with $[\tau, \xi]$ is closest to τ .

Theorem 5.3. (Nearest face of $\Pi_{nk}(G)$ in direction τ_ξ).

The nearest face of $\Pi_{nk}(G)$ in the direction τ_ξ is determined by the hyperplane

$$\sum_{i=1}^L x_{\pi_i} = \sum_{i=1}^L a_i, \quad (5.30)$$

where the sequence $\pi_1, \dots, \pi_n$ is a permutation of $J_n = \{1, \dots, n\}$ such that

$$\xi_{\pi_1} \leq \dots \leq \xi_{\pi_n}, \quad (5.31)$$

and L is determined by

$$\rho_L = \min_{l \in J_n} \rho_l, \quad (5.32)$$

where

$$\rho_l = \sum_{i=1}^l (a_i - \tau_{\pi_i}) / \sum_{i=1}^l (\xi_{\pi_i} - \tau_{\pi_i}). \quad (5.33)$$

Proof. Any point $\gamma = (\gamma_1, \dots, \gamma_n)$ on the segment $[\tau, \xi]$ can be written as

$$\gamma = (1 - \rho)\tau + \rho\xi, \quad \rho \in [0, 1]. \quad (5.34)$$

If γ belongs to the hyperplane

$$\sum_{i=1}^l x_{\pi_i} = \sum_{i=1}^l a_i,$$

then substituting (5.34) gives

$$\sum_{i=1}^l ((1 - \rho)\tau_{\pi_i} + \rho\xi_{\pi_i}) = \sum_{i=1}^l a_i,$$

from which formula (5.33) follows.

The nearest face corresponds to the smallest value of ρ_l over all $l \in J_n$, since the smallest ρ gives the intersection point closest to τ . It remains to show that the ordering (5.31) is optimal, that is, that considering only hyperplanes whose index sets satisfy (5.31) is sufficient. For index sets not satisfying (5.31), the corresponding value ρ_l is larger than the minimum over sets satisfying (5.31) for the same l , since the ordering (5.31) minimizes the denominator in (5.33) by the rearrangement inequality. Therefore the nearest face is correctly identified by (5.30)–(5.33). ■

This theorem provides an efficient algorithm for finding the nearest face: sort the components of $\xi - \tau$ in non-decreasing order to obtain the permutation $(\pi_1, \dots, \pi_n)$, then compute ρ_l for $l = 1, \dots, n$ by formula (5.33) and select L as the index minimizing ρ_l . The entire procedure requires $O(n \log n)$ operations.

The nearest face (5.30) defines a natural branching hyperplane for the branch-and-bound scheme of §5.4, as it corresponds to the first combinatorial constraint of $\Pi_{nk}(G)$ encountered when moving from the centre of the sphere toward the current relaxation solution on $S_r(\tau)$.

We now turn to the case where the objective function is quadratic:

$$\varphi(x) = x^T C x + x^T b, \quad (5.35)$$

where C is a symmetric $n \times n$ matrix and b is an n -dimensional vector.

The spherical relaxation of the optimization problem on $E_{nk}(G)$ with objective (5.35) reads:

$$\min \varphi(x), \quad x \in S_r(\tau). \quad (5.36)$$

Before solving (5.36), we address the question of whether the matrix C needs to be positive definite. In general, C may be indefinite, which complicates the analysis. However, the spherical constraint

$$\|x - \tau\|^2 = r^2$$

provides additional algebraic structure that allows us to transform any quadratic function into an equivalent one with a positive definite matrix, without changing the values of φ on $S_r(\tau)$.

Lemma 5.1. (*Reduction to positive definite case*).

For any symmetric matrix C and any scalar $\beta > 0$ such that $C + \beta I$ is positive definite, the function (5.35) and the function

$$\varphi_\beta(x) = x^T (C + \beta I) x + x^T b$$

differ on $S_r(\tau)$ by the constant

$$\beta(r^2 + \|\tau\|^2 + 2x^T \tau).$$

In particular, if β is chosen such that

$$\beta \geq \sum_{c_{ii} < 0} |c_{ii}| + \sum_{i < j} |c_{ij}|,$$

then $C + \beta I$ is positive definite, and minimizing φ on $S_r(\tau)$ is equivalent to minimizing φ_β on $S_r(\tau)$ up to this constant.

Proof. On $S_r(\tau)$ we have

$$\|x - \tau\|^2 = r^2,$$

which, when expanded, yields

$$\|x\|^2 = r^2 + 2x^T \tau - \|\tau\|^2.$$

Therefore

$$x^T (C + \beta I) x = x^T C x + \beta \|x\|^2 = x^T C x + \beta (r^2 + 2x^T \tau - \|\tau\|^2).$$

The additional term $\beta(r^2 + 2x^T \tau - \|\tau\|^2)$ is a linear function of x on $S_r(\tau)$ plus a constant, and therefore does not change the location of the minimizer of φ on $S_r(\tau)$ up to a known correction. The bound on β ensuring positive definiteness follows from the Gershgorin circle theorem. ■

Thanks to Lemma 5.1, we may assume without loss of generality that C is positive definite in what follows. This ensures that the quadratic function φ is strictly convex, which simplifies the analysis of stationarity conditions and guarantees the non-degeneracy of the solution.

To find the minimum of

$$\varphi(x) = x^T C x + x^T b$$

on $S_r(\tau)$, we apply the Lagrange multiplier method. The stationarity condition is

$$\varphi'(x) = \lambda g'(x),$$

where $g(x) = \|x - \tau\|^2 - r^2 = 0$ defines the sphere $S_r(\tau)$.

Computing the gradients gives

$$2Cx + b = 2\lambda(x - \tau), \quad (5.37)$$

which together with the constraint $\|x - \tau\|^2 = r^2$ forms the system to be solved.

We now diagonalize C by an orthogonal transformation. Let $e_1, \dots, e_n$ be the orthonormal eigenvectors of C with corresponding eigenvalues $\rho_1 \leq \dots \leq \rho_n$.

Expanding $x - \tau$ and $b + 2C\tau$ in this eigenbasis:

$$x - \tau = \sum_{i=1}^n y_i e_i,$$

$$b + 2C\tau = \sum_{i=1}^n b_i e_i,$$

where $y_i = \langle x - \tau, e_i \rangle$ and $b_i = \langle b + 2C\tau, e_i \rangle$.

Substituting into (5.37) and projecting onto e_i gives

$$2\rho_i y_i + b_i = 2\lambda y_i,$$

from which

$$y_i = \frac{b_i}{2(\lambda^* - \rho_i)} \quad (5.38)$$

provided $\lambda \neq \rho_i$.

Substituting (5.38) into the constraint

$$\sum_{i=1}^n y_i^2 = r^2$$

yields the secular equation

$$g(\lambda) = \sum_{i=1}^n \frac{b_i^2}{4(\lambda - \rho_i)^2} = r^2. \quad (5.39)$$

The secular equation (5.39) has at most $2n$ solutions – at most two on each interval between consecutive eigenvalues (one on each side of the minimum of g on that interval), found by bisection or Newton's method. Each solution λ_k^* can be found efficiently on its interval I_k by a bisection or Newton method, exploiting the convexity of g on I_k to guarantee convergence. Once λ_k^* is found, the corresponding stationary point x_k^* is recovered from (5.38):

$$x_k^* = \tau + \sum_{i=1}^n \frac{b_i e_i}{2(\lambda_k^* - \rho_i)}. \quad (5.40)$$

The global minimizer of φ on $S_r(\tau)$ is then found by evaluating φ at each of the at most $2n$ stationary points x_k^* and selecting the one with the smallest value. This procedure is computationally very efficient: the dominant cost is the eigendecomposition of C , which requires $O(n^3)$ operations, after which each stationary point is found in $O(n)$ operations.

The minimum value

$$z_s = \min_k \varphi(x_k^*)$$

provides the lower bound on z^* from the spherical relaxation.

Combined with the lower bound z_p from §5.2, the combined bound $z_l = \max\{z_p, z_s\}$ is computed at negligible additional cost once both relaxations have been solved.

The upper bound is again obtained by finding the nearest vertex of $E_{nk}(G)$ to each stationary point x_k^* using the sorting procedure of Lemma 4.3 (conditions (4.11)–(4.12), cost $O(n \log n)$) and evaluating φ at these vertices. The best (smallest) value among all candidates gives the upper bound z_u , completing the two-sided estimate $z_l \leq z^* \leq z_u$.

The sufficient optimality conditions of Chapter 4 apply here as well. By Theorem 4.2, if a point $x^* \in E_{nk}(G)$ satisfies conditions (4.15)–(4.16) with respect to the convex extension φ of the objective function, then x^* is a global minimizer of φ on $E_{nk}(G)$. In the quadratic case, the gradient $\varphi'(x^*) = 2Cx^* + b$ is readily computed, and the verification of conditions (4.15)–(4.16) requires only a comparison of orderings, as described in §5.2. Whenever this check succeeds for a candidate vertex found during the spherical relaxation, the algorithm terminates immediately with a certified global solution.

When the original problem includes functional constraints, the spherical relaxation (5.28) becomes

$$\min f(x), \quad x \in S_r(\tau), \quad g_i(x) \leq 0, \quad i \in J_p, \quad (5.41)$$

and the explicit solution via eigenvalue decomposition is no longer directly available. However, the structure of the unconstrained spherical relaxation can still be exploited effectively.

In the quadratic case, the unconstrained spherical relaxation (5.36) yields at most $2n$ stationary points $x_1^*, \dots, x_{2n}^*$ on $S_r(\tau)$, all of which can be computed explicitly by the procedure described above. Among these, those satisfying all constraints $g_i(x_k^*) \leq 0$ are feasible for (5.41), and the smallest value of φ among the feasible stationary points provides a lower bound for (5.41). If none of the stationary points is feasible, the lower bound z_s from the unconstrained relaxation remains valid for the original problem on $E_{nk}(G)$, since $E_{nk}(G) \subset S_r(\tau)$ and the constraints are assumed to be satisfied on $E_{nk}(G)$.

For each feasible stationary point x_k^* , the nearest vertex of $E_{nk}(G)$ to x_k^* is found by the sorting procedure of Lemma 4.3 and provides an upper bound candidate. The best upper bound among all candidates gives z_u . Together with the lower bound $z_l = \max\{z_p, z_s\}$, this yields the two-sided estimate $z_l \leq z^* \leq z_u$ as before.

For a general convex objective f , the unconstrained spherical relaxation

$$\min_{x \in S_r(\tau)} f(x)$$

can be solved by gradient projection methods or other methods for optimization on smooth manifolds. The resulting solution x_{s^*} may or may not satisfy the constraints, but in either case the nearest feasible vertex of $E_{nk}(G)$ provides an upper bound, and the value $z_s = f(x_{s^*})$ remains a lower bound for the original problem.

We conclude this discussion with a remark on the use of quadratic approximations, which extends the applicability of the spherical relaxation to general smooth objective functions.

Suppose f is a smooth but not necessarily quadratic convex function. At any current point $x^{(l)}$, we can construct the quadratic approximation

$$\begin{aligned} q^{(l)}(x) = & f(x^{(l)}) + \langle f'(x^{(l)}), x - x^{(l)} \rangle + \\ & + (x - x^{(l)})^T H^{(l)} (x - x^{(l)}) / 2, \end{aligned} \quad (5.42)$$

where $H^{(l)}$ is the Hessian $f''(x^{(l)})$ or its positive definite approximation.

Minimizing $q^{(l)}(x)$ on $S_r(\tau)$ is a quadratic problem of the form (5.36), which is solved explicitly by the eigenvalue procedure described above. The minimizer $x_s^{(l)}$ of $q^{(l)}$ on $S_r(\tau)$ serves as a candidate solution; the nearest vertex of $E_{nk}(G)$ to $x_s^{(l)}$ provides an upper bound for f on $E_{nk}(G)$, which can be improved iteratively as $x^{(l)}$ is updated.

This quadratic approximation approach has several attractive features. First, the dominant computational cost at each iteration is the eigen decomposition of $H^{(l)}$, which is the same cost as in the direct quadratic case. Second, the approximation quality improves as $x^{(l)}$ approaches the optimum, since the Hessian captures the local curvature of f more accurately near the solution. Third, if $H^{(l)}$ is chosen as a quasi-Newton approximation (for example, a BFGS update), the eigendecomposition can be maintained efficiently across iterations using rank-one update formulas.

The spherical relaxation of the quadratic approximation (5.42) thus provides a sequence of improving lower bounds

$$z_s^{(l)} = q^{(l)}(x_s^{(l)})$$

and upper bounds $z_u^{(l)} = f(\bar{x}^{(l)})$, where $\bar{x}^{(l)}$ is the nearest vertex of $E_{nk}(G)$ to $x_s^{(l)}$.

As $l \rightarrow \infty$, both bounds converge to z^* , providing a complete iterative optimization scheme for general smooth objectives on multipermutation combinatorial point configurations.

We now specialize the spherical relaxation to the two-level Boolean configuration $B_2^n = \{g_1, g_2\}^n$. As established in §3.7 (formula (3.22)), B_2^n admits the polyhedral-spherical representation

$$B_2^n = [g_1, g_2]^n \bigcap S_r(\tau),$$

where the hypersphere has centre

$$\tau = \left(\frac{g_1 + g_2}{2}, \dots, \frac{g_1 + g_2}{2} \right)$$

and radius

$$r = \frac{g_2 - g_1}{2} \sqrt{n}.$$

The general framework of this section applies directly: the spherical relaxation (5.28) takes the form $\min f(x)$, $x \in S_r(\tau)$, providing the lower bound $z_s \leq z^*$ since $B_2^n \subset S_r(\tau)$. For $k > 2$, the general Boolean configuration B_k^n is not spherically located (Chapter 3, §3.7), so the spherical relaxation applies to its two-level subcase $B_2^n \subset B_k^n$ only.

For a quadratic objective (5.35), the spherical relaxation on B_2^n is solved by exactly the same eigenvalue procedure as for $E_{nk}(G)$: Lemma 5.1 reduces to the positive definite case, the secular equation (5.39) is formed with the eigenvectors and eigenvalues of C and the parameters

$$r = \frac{g_2 - g_1}{2} \sqrt{n}$$

and centre c replacing τ , and the stationary points are recovered from (5.40).

By the convexity of $g(\lambda)$ on each interval between consecutive eigenvalues, the secular equation has at most two solutions per interval, found by bisection or Newton's method. The dominant cost remains $O(n^3)$ for the eigendecomposition of C .

The decisive difference from $E_{nk}(G)$ appears in the nearest-vertex computations. Finding the nearest vertex of B_2^n to a stationary point x_k^* is solved by Lemma 4.13 according to condition (4.30): for each coordinate $i = 1, \dots, n$ independently, the nearest element of $\{g_1, g_2\}$ to $(x_k^*)_i$ is found by a single comparison, requiring $O(1)$ per coordinate and $O(n)$ overall. This contrasts with the $O(n \log n)$ sorting required for $E_{nk}(G)$ via Lemma 4.3.

As noted in §4.6, Lemma 4.13 replaces the assignment problem of Lemma 4.3 by n independent binary choices, cleanly demonstrating the structural gain from coordinate independence. The combined lower bound (5.29) and the two-sided estimate $z_l \leq z^* \leq z_u$ apply to B_2^n without modification. The sufficient optimality condition is Theorem 4.12 via condition (4.32), checked coordinate wise in $O(n)$, and the fixed-point characterization of Theorem 4.15 (condition (4.34)) applies for the strongly convex case.

The quadratic approximation scheme (5.42) extends to B_2^n without modification, with all nearest-vertex computations replaced by the coordinatewise procedure of Lemma 4.13. The sequence of bounds $z_s^{(l)}$ and $z_u^{(l)}$ converges to z^* as before, with the reduced $O(n)$ cost per nearest-vertex computation reflecting the coordinate independence of B_2^n .

Thus, the spherical relaxation approach is developed at two levels. At the general level, it applies to any polyhedral-spherical CPC and provides the lower bound $z_s \leq z^*$, whose combination with z_p from §5.2 gives the sharper combined bound (5.29). Theorem 5.2 establishes that neither relaxation dominates the other in general, making their combination essential. At the specific level, the quadratic case admits a complete explicit solution via the secular equation (5.39) and the eigen decomposition of C . Theorem 5.3 provides the nearest-face computation central to the branching step of §5.4. For multipermutation point configurations $E_{nk}(G)$, the nearest-vertex computation costs $O(n \log n)$ via Lemma 4.3; for Boolean configurations B_2^n , it costs $O(n)$ via Lemma 4.13. The methods of this section and §5.2 are combined into a complete branch-and-bound algorithm in §5.4.

§5.4. Dimension Reduction and Branch-and-Bound Scheme for Multipermutation Point Configurations

The relaxation approaches of §5.2 and §5.3 provide the lower bound $z_l = \max\{z_p, z_s\}$ and an upper bound z_u obtained by evaluating the objective at the nearest vertex of E to the relaxation solution. When the gap between z_u and z_l exceeds the prescribed tolerance ε , the search must be continued by decomposing the problem into subproblems of smaller size. This decomposition is achieved by two complementary mechanisms: branching, which partitions E into subsets — for multipermutation configurations by intersecting with a family of hyperplanes, and for Boolean configurations by fixing individual coordinates and dimension reduction, which exploits the resulting equality constraints to reduce the dimension of the relaxation subproblems at each subsequent level. Together, branching and dimension reduction form the backbone of the branch-and-bound scheme developed in this section.

The branch-and-bound scheme is not an independent method standing alongside the relaxations of §5.2 and §5.3. It is the organizational framework that combines them into a complete algorithm for global optimization on a vertex-located CPC E . The lower bounds at each node come from the polyhedral and spheri-

cal relaxations. The upper bounds come from the nearest-vertex procedure. The branching rules come from the hyperplane decomposition of E described in Chapter 3. The dimension reduction makes the relaxation subproblems at deeper levels of the tree significantly cheaper to solve. The sufficient optimality conditions of Chapter 4 provide a mechanism for early termination whenever a globally optimal solution is found before the tree is fully explored.

The general principle of dimension reduction is as follows. Each branching step adds an equality constraint that restricts the problem to an affine subspace of dimension $n - 1$. After j branching steps the subproblem lives in a space of dimension $n - j$.

For vertex-located CPCs admitting a polyhedral-spherical representation (5.11), this requirement is automatically satisfied: the intersection of the polytope P with a branching hyperplane is again a polytope, and the intersection of the hypersphere $S_r(\tau)$ with the same affine subspace is a lower-dimensional hypersphere. The relaxation subproblem at each node therefore has exactly the same structure as the original problem but in reduced dimension, and all tools developed in §5.2 and §5.3 apply without modification.

The nearest-vertex computation plays a dual role in the algorithm. On the one hand, it provides the upper bound z_u at each node: the objective is evaluated at the nearest vertex $\tilde{x}$ to the relaxation solution, and z_u is updated whenever $\varphi(\tilde{x}) < z_u$.

On the other hand, the same candidate vertex $\tilde{x}$ is immediately subjected to the sufficient optimality check of Chapter 4. For vertex-located CPCs, this check reduces to a comparison of orderings of the gradient and the coordinate values of $\tilde{x}$, and it costs $O(n \log n)$ for multipermutation configurations and $O(n)$ for Boolean configurations. Whenever the check succeeds, the algorithm terminates immediately with $\tilde{x}$ as a certified global solution, without exhausting the branching tree. This dual role is what makes the nearest-vertex computation central to the efficiency of the scheme: it simultaneously tightens the gap $z_u - z_l$ from above and provides a termination criterion that can save substantial computation whenever the optimum is encountered early.

The two-sided estimate $z_l^{(j)} \leq z^* \leq z_u$ is maintained at every node of the branching tree. The lower bound $z_l^{(j)}$ is non-decreasing along every path from the root to a leaf, so the gap narrows monotonically as the tree is explored. A node is pruned whenever $z_l^{(j)} \geq z_u$, since no descendant of that node can improve upon the current best feasible solution. The combination of a tight lower bound from two complementary relaxations and a tight upper bound from the nearest-vertex procedure makes the pruning highly effective in practice, and the algorithm typically explores only a small fraction of the complete branching tree.

The two principal configuration classes realize dimension reduction in structurally different ways. For multipermutation point configurations $E_{nk}(G)$, the branching hyperplane is generally oblique with respect to the coordinate axes, and an explicit orthogonal change of variables is required to maintain the spherical structure of the subproblem.

For Boolean configurations B_k^n , the branching hyperplane is a coordinate hyperplane, and dimension reduction reduces to the direct elimination of one coordinate variable without any transformation.

We now develop the branch-and-bound scheme in detail for multipermutation point configurations $E_{nk}(G)$. Recall from §5.2 that the branching scheme adds equality constraints of the form

$$\sum_{i=1}^s x_{\pi_i} = C_{t^s} \quad (5.43)$$

to the relaxation problem at each level of the branching tree.

Each such constraint restricts the problem to an affine subspace of dimension $n - 1$, and after j branching steps the problem is restricted to an affine subspace of dimension $n - j$. In §5.2 we noted that this reduction occurs implicitly, without an explicit change of variables.

We now show how to perform this reduction explicitly by means of an orthogonal transformation, which is essential for the efficient solution of the spherical relaxation subproblem at each node.

After adding j hyperplane constraints $H_1, \dots, H_j$ of the form (5.43), the intersection

$$\Pi_{nk}(G) \cap H_1 \cap \dots \cap H_j$$

is a polytope lying in an affine subspace A_j of dimension $n - j$.

The circumsphere $S_r(\tau)$ intersected with A_j is a sphere of dimension $n - j - 1$ lying in A_j . Therefore the spherical relaxation subproblem at level j is a problem of minimizing f on an $(n - j)$ -sphere in an $(n - j)$ -dimensional space, exactly the same type of problem as the original spherical relaxation but in lower dimension.

To pass to an explicit $(n - j)$ -dimensional formulation, we apply a sequence of orthogonal transformations. We describe one step; the general step follows by induction. Suppose that at the current node we have added one hyperplane constraint

$$H : \sum_{i=1}^{l^*} x_{\pi_i} = C_{t^s}, \quad (5.44)$$

where the indices $\pi_1, \dots, \pi_{l^*}$ are determined by Theorem 5.3 as the nearest face of $\Pi_{nk}(G)$ in the direction of the current spherical relaxation solution.

Let O_1 denote the orthogonal projection of the centre τ onto H , and let U_1 be an orthogonal $n \times n$ matrix whose first row is the unit normal to H :

$$e_H = (1/\sqrt{l^*})(e_{\pi_1} + \dots + e_{\pi_{l^*}}),$$

where e_i denotes the i -th standard basis vector.

The transformation

$$y = U_1(x - O_1)$$

maps the hyperplane H to the coordinate hyperplane in $\mathbb{R}^n$, so that constraint (5.44) becomes simply $y_1 = 0$. Dropping the first coordinate reduces the problem to an $(n - 1)$ -dimensional problem in the variables $y_2, \dots, y_n$. Under this transformation, the objective function (5.35) becomes

$$\varphi_1(y) = y^T C_1 y + y^T b_1,$$

where

$$C_1 = U_1 C U_1^T, \quad b_1 = U_1 (b + 2 C O_1). \quad (5.45)$$

The matrix C_1 is again symmetric, and if C is positive definite then C_1 is also positive definite, since orthogonal transformations preserve eigenvalues. The sphere $S_r(\tau)$ is mapped to a sphere of the same radius r centred at $U_1(\tau - O_1)$, which lies in hyperplane $y_1 = 0$.

The radius of the reduced sphere is

$$r_1 = \sqrt{r^2 - \|O_1 - \tau\|^2},$$

where $\|O_1 - \tau\|$ is the distance from the centre τ to the hyperplane H .

Restricting to $y_1 = 0$ gives an $(n - 2)$ -sphere in $\mathbb{R}^{n-1}$, and the reduced problem is

$$\min_{(y_2, \dots, y_n) \in S_{r_1}(\tau_1)} \varphi_1(y_2, \dots, y_n), \quad (5.46)$$

where r_1 and τ_1 are the radius and centre of the reduced sphere, computed from r , τ , O_1 , and U_1 .

After j branching steps with hyperplanes $H_1, \dots, H_j$, the reduction is applied j times, yielding the $(n - j)$ -dimensional problem

$$\min_{(y_{j+1}, \dots, y_n) \in S_{r_j}(\tau_j)} \varphi_j(y_{j+1}, \dots, y_n), \quad (5.47)$$

where φ_j is a quadratic function with matrix

$$C_j = U_j \cdots U_1 C U_1^T \cdots U_j^T$$

and the sphere $S_{r_j}(\tau_j)$ has dimension $n - j - 1$.

Problem (5.47) has exactly the same structure as the original spherical relaxation (5.36), and is solved by the eigenvalue procedure of §5.3 applied to the reduced $(n - j) \times (n - j)$ matrix C_j . The lower bound from the reduced spherical relaxation is

$$z_{S^{(j)}} = \min_{(y_{j+1}, \dots, y_n) \in S_{r_j}(\tau_j)} \varphi_j(y_{j+1}, \dots, y_n) \quad (5.48)$$

which is tighter than $z_{S^{(j-1)}}$ since the feasible set has been further restricted.

The polyhedral relaxation subproblem at level j is handled as follows. The polytope $\Pi_{nk}(G) \cap H_1 \cap \dots \cap H_j$ is the generalized permutohedron with additional equality constraints. These constraints are simply appended to the H-representation of $\Pi_{nk}(G)$ without modifying it, and the conditional gradient method with sorting via Lemma 4.1 applies as before. The lower bound $z_p^{(j)}$ from the polyhedral relaxation at level j is at least as large as $z_p^{(j-1)}$ since the feasible set has been restricted.

The combined lower bound at level j is therefore

$$z_l^{(j)} = \max\{z_p^{(j)}, z_s^{(j)}\}, \quad (5.49)$$

which is non-decreasing as j increases. This monotone improvement of bounds as the branching tree is explored is a fundamental property that guarantees both the correctness and the efficiency of the scheme.

At each node at level j , the upper bound is obtained by finding the nearest vertex of $E_{nk}(G)$ to each stationary point $x_{[S,k]}^{(j)}$ on $S_{r_j}(\tau_j)$. By Lemma 4.3 and conditions (4.11)–(4.12), this nearest vertex is found by sorting the components of $x_{[S,k]}^{(j)}$ at cost $O(n \log n)$. The upper bound at level j is

$$z_u = \min_k \varphi(\bar{x}_{k^{(j)}}),$$

where $\bar{x}_k^{(j)}$ is the nearest vertex of $E_{nk}(G)$ to $x_{[S,k]}^{(j)}$. The two-sided estimate

$$z_l^{(j)} \leq z^* \leq z_u$$

is thus maintained at negligible additional cost once both relaxations have been solved.

Each candidate vertex $\bar{x}_k^{(j)}$ is immediately subjected to the sufficient optimality check of Theorem 4.2, conditions (4.15)–(4.16): if the gradient

$$\varphi'(\bar{x}_{k^{(j)}}) = 2C\bar{x}_{k^{(j)}} + b$$

satisfies the ordering conditions with respect to the coordinates of $\bar{x}_k^{(j)}$, then $\bar{x}_k^{(j)}$ is certified as a global minimizer of φ on $E_{nk}(G)$, and the algorithm terminates immediately. This check requires only a comparison of orderings, costs $O(n \log n)$, and can save substantial computation whenever the global optimum is encountered early in the search.

The dominant computational cost at each node at level j is the eigendecomposition of the reduced matrix C_j , which costs $O((n-j)^3)$. All remaining operations at the node cost $O(n \log n)$: the nearest vertex via Lemma 4.3 (conditions (4.11)–(4.12)), the optimality check via conditions (4.15)–(4.16), the nearest face via Theorem 5.3, and the orthogonal transformation (5.45) which costs $O(n^2)$. As j increases, the eigendecomposition cost $O((n-j)^3)$ decreases rapidly, so the deeper levels of the tree are significantly cheaper than the root.

We now describe the complete branch-and-bound algorithm for global optimization on $E_{nk}(G)$. The algorithm maintains a list of active nodes, each defined by a set of branching constraints, and a global upper bound z_u updated whenever a feasible point with a lower objective value is found.

The algorithm begins at the root node (*level* $j = 0$). The polyhedral relaxation (5.13) is solved by the conditional gradient method of §5.2, giving the lower bound $z_p^{(0)}$ and the relaxation solution $x_p^{(0)} \in \Pi_{nk}(G)$. The spherical relaxation (5.36) is solved by the eigenvalue procedure of §5.3, giving the lower bound $z_s^{(0)}$ and the set of stationary points $\{x_{\{S,k\}}^{(0)}\}$ on $S_r(\tau)$. The combined lower bound at the root is

$$z_l^{(0)} = \max\{z_p^{(0)}, z_s^{(0)}\}.$$

The upper bound at the root is

$$z_{u^{(0)}} = \min_k \varphi(\bar{x}_{k^{(0)}}),$$

where $\bar{x}_k^{(0)}$ is the nearest vertex of $E_{nk}(G)$ to $x_{\{S,k\}}^{(0)}$, found by the sorting procedure of Lemma 4.3.

The sufficient optimality conditions (4.15)–(4.16) are checked at each $\bar{x}_k^{(0)}$; if satisfied, the algorithm terminates with a certified global solution.

If $z_u^{(0)} - z_l^{(0)} \leq \varepsilon$, the algorithm terminates with the current best feasible point as the ε -optimal solution. Otherwise, branching is performed.

At each node at level j , the branching step proceeds as follows. From the spherical relaxation solution, the nearest face of $\Pi_{nk}(G)$ in the direction $\tau x_S^{(j)}$ is identified using Theorem 5.3. This face determines the branching hyperplane H_{j+1} of the form (5.43). The current node is split into child nodes corresponding to the different hyperplanes in the family, each defining a subproblem on a smaller subset of $E_{nk}(G)$.

For each child node, the orthogonal transformation U_{j+1} is computed from the normal to H_{j+1} by formula (5.45), and the reduced problem (5.47) in dimension $n - j - 1$ is formulated. Both relaxations are solved for the reduced problem, giving the bounds $z_p^{(j+1)}$ and $z_s^{(j+1)}$ and the combined bound

$$z_l^{(j+1)} = \max\{z_p^{(j+1)}, z_s^{(j+1)}\}.$$

The nearest vertex of $E_{nk}(G)$ to each stationary point is found by the sorting procedure of Lemma 4.3 and provides a candidate for updating z_u and for the sufficient optimality check. A child node is pruned if $z_l^{(j+1)} \geq z_u$, since the subproblem at this node cannot contain a solution better than the current best.

The upper bound z_u is updated whenever a feasible point $x \in E_{nk}(G)$ with $\varphi(x) < z_u$ is found, either as a nearest vertex to a relaxation solution or by the sufficient optimality check. The algorithm terminates when the list of active nodes is empty, at which point the current best feasible point is the global ε -optimal solution, or when conditions (4.15)–(4.16) are satisfied at some candidate point, certifying exact global optimality.

Theorem 5.4. (*Correctness and finite termination*).

The branch-and-bound algorithm described above has the following properties:

- (i) At every node of the branching tree, the combined lower bound $z_l^{(j)}$ satisfies $z_l^{(j)} \leq z^*$, and the upper bound z_u satisfies $z_u \geq z^*$.
- (ii) The lower bounds $z_l^{(j)}$ are non-decreasing along any path from the root to a leaf of the branching tree.
- (iii) The algorithm terminates in a finite number of steps with either an ε -optimal solution or an exactly optimal solution certified by conditions (4.15)–(4.16).

Proof. Property (i) follows from the validity of the polyhedral and spherical relaxations as lower bounds, established in §5.2 and §5.3, and from the fact that z_u is always the value of φ at a feasible point of $E_{nk}(G)$.

Property (ii) follows from the monotonicity of the relaxation bounds: adding equality constraints to the polyhedral relaxation can only increase z_p , and restricting the sphere to a lower-dimensional subsphere can only increase z_s , since the feasible set shrinks at each level.

Property (iii) follows from the finiteness of $E_{nk}(G)$: the branching tree has depth at most $n - 1$ (after $n - 1$ branching steps each node contains at most one element of $E_{nk}(G)$ by Theorem 3.6 of Chapter 3), and the number of nodes at each level is bounded by the number of hyperplanes in the corresponding family.

Therefore the total number of nodes is finite and the algorithm must terminate. If it terminates by exhausting the branching tree, the current best feasible point is ε -optimal. If it terminates by satisfying conditions (4.15)–(4.16), the solution is exactly optimal. ■

The theorem guarantees that the algorithm is both correct and finite. In practice, the pruning of nodes by the combined lower bound

$$z_l^{(j)} = \max\{z_p^{(j)}, z_s^{(j)}\}$$

is highly effective, and the algorithm typically explores only a small fraction of the complete branching tree. The early termination criterion based on conditions (4.15)–(4.16) further reduces the computational effort whenever the global optimum satisfies the gradient ordering condition.

The efficiency of the algorithm depends critically on the quality of the lower bounds at each node. The combined bound $z_l^{(j)} = \max\{z_p^{(j)}, z_s^{(j)}\}$ is in general significantly tighter than either $z_p^{(j)}$ or $z_s^{(j)}$ alone, as established by Theorem 5.2. When the unconstrained minimiser $\hat{x}$ lies in the interior of $\Pi_{nk}(G)$, the spherical relaxation provides a tighter bound, while when $\hat{x}$ lies outside $\Pi_{nk}(G)$, the polyhedral relaxation is more informative. By using both at every node, the algorithm automatically adapts to the geometry of each subproblem without requiring any prior knowledge of the location of $\hat{x}$.

The vertex-located property of $E_{nk}(G)$ is what makes the nearest-vertex computation both efficient and informative. Because $E_{nk}(G) = \text{vert } \Pi_{nk}(G)$, the nearest

vertex to any point is found by a single sorting operation of cost $O(n \log n)$ via Lemma 4.3, without solving a general projection problem. Moreover, the sorted order of the coordinates of the nearest vertex is precisely what is needed to check the sufficient optimality conditions (4.15)–(4.16), so the sorting serves both purposes simultaneously. This structural advantage is not available for non-vertex-located configurations, where the nearest-vertex problem would require a more expensive combinatorial procedure.

The complete algorithm integrates all the components developed in this chapter: the polyhedral-spherical representation of $E_{nk}(G)$ from Chapter 3, the convex extension theory and optimality conditions from Chapter 4, the polyhedral relaxation and conditional gradient method from §5.2, the spherical relaxation and eigenvalue solution from §5.3, and the dimension reduction by orthogonal transformation described in this section. The analogous scheme for unbounded partial multipermutation configurations B_k^n , where coordinate independence replaces the permutation structure and eliminates the need for orthogonal transformations, is developed in §5.5.

§5.5. Dimension Reduction and Branch-and-Bound Scheme for Unbounded Partial Multipermutation Point Configurations

The branch-and-bound scheme developed in §5.4 for multipermutation configurations $E_{nk}(G)$ relied on several computationally nontrivial components: the H -representation of the generalized permutohedron $\Pi_{nk}(G)$ required up to 2^n inequalities, the auxiliary linear subproblem demanded a sorting procedure of cost $O(n \log n)$, the dimension reduction at each node required an explicit orthogonal transformation of cost $O(n^2)$, and the nearest-vertex computation for the sufficient optimality check required sorting at cost $O(n \log n)$. The present section shows that all of these constructions simplify fundamentally for unbounded partial multipermutation configurations $B_k^n = G^n$, where each coordinate is chosen independently from the numerical realization $G = \{g_1 < \dots < g_k\}$.

The decisive structural feature is coordinate independence. Since no permutation or multiplicity constraint links the coordinates of B_k^n , every auxiliary problem decomposes into n independent one-dimensional subproblems, each solved in $O(1)$ operations. As a consequence: the linear subproblem is solved coordinatewise in $O(n)$ rather than $O(n \log n)$; the nearest vertex is found by coordinatewise rounding in $O(kn)$ rather than $O(n \log n)$; the nearest face is identified in $O(n)$

rather than $O(n \log n)$; the dimension reduction costs $O(1)$ per step rather than $O(n^2)$; and the sufficient optimality check costs $O(n)$ rather than $O(n \log n)$. The complete theory of §§5.2–5.4 carries over without modification, with every sorting-based procedure replaced by a coordinatewise selection.

Recall from §3.7 that B_k^n admits the polyhedral-spherical representation

$$B_k^n = Q_k^n \cap S_r(\tau)$$

(formulas (3.20)–(3.22)), where

$$Q_k^n = \{x \in \mathbb{R}^n : g_1 \leq x_i \leq g_k, i \in J_n\}$$

is the n -dimensional hyperrectangle, and $S_r(\tau)$ is the circumscribed hypersphere with centre

$$\tau = ((g_1 + g_k)/2, \dots, (g_1 + g_k)/2) \quad (5.50)$$

and radius

$$r = \sqrt{n} (g_k - g_1) / 2. \quad (5.51)$$

The vertex-located property

$$B_k^n = \text{vert } Q_k^n$$

holds exactly when $k = 2$.

For general $k \geq 2$, the configuration B_k^n consists of the k^n lattice points of a regular grid inscribed in Q_k^n , all of which lie on $S_r(\tau)$. Only the 2^n extreme vertices of Q_k^n are actual corners of the hyperrectangle. The convex extensions and extremal theorems of Chapter 4 hold for all $k \geq 2$, with the Boolean case $k = 2$ enjoying the additional vertex-located property.

The polyhedral relaxation of the optimization problem on B_k^n is obtained by replacing the constraint $x \in B_k^n$ by $x \in Q_k^n$:

$$\min f(x), \quad x \in Q_k^n, \quad g_i(x) \leq 0, \quad i \in J_r. \quad (5.52)$$

Since f and all g_i are convex on Q_k^n by the framework of §5.1, problem (5.52) is a convex programming problem whose optimal value $z_p^* \leq z^*$ is a valid lower bound.

The auxiliary linear subproblem at each iteration of the conditional gradient method is: minimize $\langle c, y \rangle$ over Q_k^n . By Lemma 4.12 and condition (4.29), the solution is determined coordinatewise:

$$y_i^* = \begin{cases} g_1 & \text{when } c_i \geq 0; \\ g_k & \text{when } c_i < 0. \end{cases}$$

No ordering of coordinates is needed, and the total cost is $O(n)$. This replaces the $O(n \log n)$ sorting procedure of §5.2 for $E_{nk}(G)$.

The spherical relaxation of the problem on B_k^n is obtained by replacing the constraint $x \in B_k^n$ by $x \in S_r(\tau)$:

$$\min f(x), \quad x \in S_r(\tau), \quad (5.53)$$

where the centre τ and radius r are given by (5.50)–(5.51). Since $B_k^n \subset S_r(\tau)$ by formula (3.22), the minimum z_s of (5.53) satisfies $z_s \leq z^*$, providing a second valid lower bound.

For a quadratic objective $\varphi(x) = x^T Cx + x^T b$, the spherical relaxation (5.53) is solved by the eigenvalue procedure of §5.3 without any modification. The secular equation (5.39), the formulas for stationary points (5.40), and the algorithm for finding the global minimum on $S_r(\tau)$ apply identically, with the values (5.50)–(5.51) substituted for the centre and radius. The at most $2n$ stationary points are enumerated, the global minimum among them gives z_s , and the combined bound $z_l = \max\{z_p, z_s\}$ of (5.29) is used throughout the branch-and-bound scheme.

The relative sharpness of z_p and z_s is governed by the same geometric principle as Theorem 5.2: when the unconstrained minimiser $\hat{x}$ lies in the interior of Q_k^n , the spherical relaxation is typically sharper; when $\hat{x}$ lies outside Q_k^n , the polyhedral relaxation is typically sharper since the hyperrectangle directly captures the combinatorial geometry of B_k^n .

The gradient-based lower bound for convex differentiable φ is given by Theorem 4.11 and condition (4.31). For any $x \in X \supseteq B_k^n$, the lower bound is computed by applying condition (4.29) to the gradient components $\partial\varphi(x)/\partial x_i$, giving

$$z_l = \varphi(x^{(l)}) - \langle \varphi'(x^{(l)}), x^{(l)} \rangle + \langle \langle \varphi'(x^{(l)}), y \rangle, y \rangle \leq z^* \quad (5.54)$$

The last term is evaluated in $O(n)$ operations by condition (4.29). For strongly convex φ , the sharper bound of Theorem 4.14 and condition (4.33) gives

$$\begin{aligned} \min_{y \in B_k^n} \varphi(y) &\geq \varphi(x) - \langle \varphi'(x), x \rangle + \rho \|x\|^2 - \left(\frac{1}{4\rho} \right) \|\varphi'(x)\|^2 + \\ &\quad + \rho \cdot \min_{y \in B_k^n} \|y - c\|^2, \end{aligned}$$

where $c = x - (1/(2\rho))\varphi'(x)$ and the last term is found by condition (4.30) in $O(kn)$ operations.

The nearest vertex of B_k^n to a given point $x^0 \in \mathbb{R}^n$ is found by Lemma 4.13 and condition (4.30): for each coordinate i independently, x_i^* equals the element of G closest to x_i^0 in the Euclidean metric. This coordinatewise rounding requires $O(kn)$ operations, replacing the $O(n \log n)$ sorting of §5.2.

As in §5.4, the nearest-vertex computation plays a dual role. On the one hand, it provides the upper bound: the objective is evaluated at the nearest vertex $\bar{x}$ to each relaxation solution, and z_u is updated whenever $\varphi(\bar{x}) < z_u$. The two-sided estimate

$$z_l \leq z^* \leq z_u \quad (5.55)$$

is thus maintained at every node. On the other hand, the same candidate vertex $\bar{x}$ is immediately subjected to the sufficient optimality check of Theorem 4.12, condition (4.32): for each coordinate i , the check verifies that

$$\frac{\partial \varphi(\bar{x})}{\partial x_i} \begin{cases} \geq 0 & \text{whenever } \bar{x}_i = g_1; \\ < 0 & \text{whenever } \bar{x}_i = g_k. \end{cases}$$

The geometric meaning is transparent: the gradient at $\bar{x}$ points away from the interior of Q_k^n at every active face, so no descent direction into Q_k^n exists and $\bar{x}$ is already optimal over B_k^n . This check costs $O(n)$ and can terminate the algorithm immediately with a certified global solution.

The vertex-located property of B_k^n (when $k = 2$) is what makes this optimality check particularly efficient. Because every point of B_k^n is a vertex of $Q_k^n = [g_1, g_2]^n$, the rounding step of condition (4.30) delivers a candidate that is both a feasible point and a vertex simultaneously. The subsequent gradient check (4.32) requires only n comparisons, one per coordinate. For general $k \geq 2$, the same rounding delivers the nearest grid point, but the optimality check (4.32) applies only at the extreme values g_1 and g_k , and non-extreme grid points require a separate analysis via the gradient bounds of Theorems 4.11–4.14.

The fixed-point condition of Theorem 4.15 and condition (4.34) provides the corresponding sufficient optimality criterion for the strongly convex case: a point $x^* \in B_k^n$ is a minimizer of φ if it is the nearest point of B_k^n to the shifted centre

$$c^* = x^* - (1/(2\rho))\varphi'(x^*).$$

This check is performed by condition (4.30) at cost $O(kn)$ and subsumes condition (4.32) as a special case for the convex setting.

The nearest face of Q_k^n to the current relaxation solution $\tilde{x}$ is identified coordinatewise. For each i , compute

$$\begin{aligned} d_i^+ &= |\tilde{x}_i - g_k|, \\ d_i^- &= |\tilde{x}_i - g_1|. \end{aligned}$$

Let

$$i^* = \arg \min_i (\min\{d_i^+, d_i^-\}) .$$

The nearest face is

$$y_i^* = \begin{cases} g_1 & \text{if } d_i^- \leq d_i^+; \\ g_k & \text{otherwise.} \end{cases} \quad (5.56)$$

This identification requires $O(n)$ operations, replacing the $O(n \log n)$ procedure of Theorem 5.3 used for $E_{nk}(G)$.

Once the nearest face F_{i^*} is identified, the dimension reduction is immediate. Setting coordinate x_{i^*} to its boundary value and substituting into the objective and constraints yields a problem of the same form in dimension $n - 1$, with the reduced configuration $B_k^{n-1} = G^{n-1}$. No orthogonal transformation is needed: the hyperrectangle Q_k^{n-1} is the projection of Q_k^n onto the remaining $n - 1$ coordinates, and the reduced sphere $S_r(\tau')$ is obtained from $S_r(\tau)$ by removing the fixed coordinate. The reduced centre, radius, and objective are computed in $O(n)$ operations by direct substitution. This contrasts with the orthogonal transformation (5.45) for $E_{n_k}(G)$, which costs $O(n^2)$ per step.

After j branching steps fixing coordinates $i_1, \dots, i_j$, the reduced problem is a $(n - j)$ -dimensional quadratic problem over $B_k^{n-1} = G^{n-1}$ with sphere of radius

$$r_j = \sqrt{n-j} (g_k - g_1) / 2.$$

The dominant cost per node at level j is the eigen decomposition of the reduced matrix $C^{(i)}$ of size $(n - j) \times (n - j)$, costing $O((n - j)^3)$. All remaining operations at the node cost $O(n)$: the nearest vertex by condition (4.30), the optimality check by condition (4.32), the nearest face by formula (5.56), and the dimension reduction by direct substitution. As j increases, the eigen decomposition cost decreases rapidly, and the deeper levels of the tree are significantly cheaper than the root.

The branch-and-bound algorithm for B_k^n follows the same organizational framework as §5.4, with all auxiliary procedures replaced by their $O(n)$ counterparts. At each node at level j , the current subproblem is defined by fixing j coordinates to boundary values from G , yielding the reduced configuration $B_k^{n-1} = G^{n-1}$. Both relaxations (5.52) and (5.53) are solved for the reduced problem in dimension $n - j$, giving the bounds $z_p^{(j)}$ and $z_s^{(j)}$ and the combined lower bound

$$z_l^{(j)} = \max\{z_p^{(j)}, z_s^{(j)}\}. \quad (5.57)$$

The upper bound z_u is updated by the nearest vertex found by condition (4.30), and condition (4.32) is checked at each candidate vertex. A node is pruned whenever $z_l^{(j)} \geq z_u$.

The branching step selects the coordinate i^* identified by (5.56) and creates two child nodes corresponding to $x_{i^*} = g_1$ and $x_{i^*} = g_k$. For the Boolean case $k = 2$, this binary branching is the only type available, and each step reduces the dimension by exactly one. For general $k \geq 2$, one may alternatively branch into k child nodes, one for each value $g_1, \dots, g_k$ of coordinate i^* producing a k -ary branching tree of depth at most n and total nodes bounded by $k^n = |B_k^n|$.

The correctness and finite termination of the scheme follow from the same argument as Theorem 5.4: the lower bounds $z_l^{(j)}$ are valid and non-decreasing along every path, the upper bounds are values of ϕ at feasible points, the branching tree has finite depth n with at most k^n leaves, and condition (4.32) provides a mechanism for early termination.

The branch-and-bound algorithm for B_k^n terminates in at most k^n steps with an ϵ -optimal solution. If condition (4.32) is satisfied at some candidate vertex, the algorithm terminates with an exactly optimal solution certified by Theorem 4.12 of Chapter 4.

The Boolean case B_2^n deserves separate emphasis. Here $G = \{g_1, g_2\}$, the hyperrectangle is the hypercube $[g_1, g_2]^n$, and every point of B_k^n is a vertex of $[g_1, g_2]^n$: the vertex-located property

$$B_k^n = \text{vert } [g_1, g_2]^n$$

holds.

The nearest face identification (5.56) reduces to comparing $|\tilde{x}_i - g_2|$ and $|\tilde{x}_i - g_1|$ for each coordinate, deciding whether $\tilde{x}_i$ is closer to g_2 or to g_1 . The sufficient optimality condition (4.32) becomes:

$$\frac{\partial \phi(x)}{\partial x_i} \begin{cases} \geq 0 & \text{whenever } \bar{x}_i = g_1; \\ < 0 & \text{whenever } \bar{x}_i = g_2. \end{cases}$$

The entire branching tree is a binary tree of depth n with at most 2^n leaves, corresponding to the 2^n vertices of the hypercube.

The vertex-located property of B_2^n means that condition (4.30) delivers a true vertex of B_2^n at every node, and the optimality check (4.32) applies directly without any further rounding. This is the same structural advantage as for $E_{nk}(G)$ in §5.4: the nearest vertex is obtained in $O(n)$ by a single coordinatewise comparison instead of sorting, and the gradient check follows at the same cost. The two operations together cost $O(n)$ at every node, compared to $O(n \log n)$ for $E_{nk}(G)$.

The central symmetry of B_2^n noted in §3.7 can be exploited to reduce the search further. Every vertex $x \in B_2^n$ has an antipodal partner also in B_2^n . If a node with lower bound $z_i^{(j)}$ has been pruned, its antipodal node can often be pruned immediately by the same bound, reducing the effective tree size by a factor of up to two. This symmetry exploitation has no analogue for $E_{nk}(G)$.

The methods developed in this section provide a complete framework for global optimization on B_k^n and its Boolean special case B_2^n . The framework inherits all structural advantages of §§5.2–5.4 while exploiting coordinate independence to replace every $O(n \log n)$ sorting procedure by an $O(n)$ coordinatewise selection, and the orthogonal transformation of cost $O(n^2)$ by direct coordinate elimination at cost $O(1)$. The dominant cost per node remains the eigen decomposition of the reduced matrix at cost $O((n-j)^3)$, identical in structure to §5.4 but with all auxiliary steps accelerated by a factor of $\Omega(\log n)$. The nearest-vertex computation plays the same dual role as in §5.4: it provides both the upper bound and the input for the sufficient optimality check, which for B_2^n costs only $O(n)$ per node.

§5.6. A Penalty-Function Method for Optimization on Polyhedral-Spherical Configurations

The methods developed in §§5.2–5.5 exploit the polyhedral-spherical representation $E = P \cap S_r(\tau)$ in two principal ways: by solving a polyhedral relaxation on the polytope P (§5.2) and a spherical relaxation on the circumsphere $S_r(\tau)$ (§5.3), and by combining these relaxations in branch-and-bound schemes (§§5.4–5.5). Both relaxations are continuous and yield lower bounds on the optimum. The present section proposes a complementary approach that converts the membership constraint $x \in S_r(\tau)$ into a penalty term and solves a sequence of continuous optimization problems on the polytope P alone. Because every point of E lies on $S_r(\tau)$, the penalty forces iterates towards the vertices of P that lie on the sphere, recovering elements of E in the limit.

We consider the general optimization problem on a polyhedral-spherical configuration:

$$\min f(x), \quad x \in E = P \cap S_r(\tau), \quad (5.58)$$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is continuously differentiable, P is the combinatorial polytope associated with E (either the generalised permutohedron $\Pi_{nk}(G)$ or the hyperrectangle Q_k^n), and $S_r(\tau)$ is the circumsphere of E with centre τ and radius r determined by formulas (5.18) for $E_{nk}(G)$ and (5.50)–(5.51) for B_k^n .

Problem (5.58) covers all settings treated in §§5.2–5.5 and does not restrict f to be linear.

The central observation is that membership in $S_r(\tau)$ is characterised by a single smooth equation:

$$\|x - \tau\|^2 - r^2 = 0$$

if and only if $x \in S_r(\tau)$. Squaring the left-hand side yields a non-negative function that equals zero precisely on $S_r(\tau)$. This motivates introducing the spherical penalty function

$$\Psi(\lambda, x) = f(x) + \lambda(\|x - \tau\|^2 - r^2)^2, \quad x \in P, \quad \lambda > 0. \quad (5.59)$$

The penalty term $\lambda(\|x - \tau\|^2 - r^2)^2$ vanishes if and only if x lies on $S_r(\tau)$, and grows without bound as x moves away from the sphere. For any fixed $\lambda > 0$, the subproblem

$$\min \Psi(\lambda, x), \quad x \in P \quad (5.60)$$

is a continuous optimization problem on the polytope P , to which standard gradient-based or Frank–Wolfe methods are applicable. As λ increases, the set of points of P that are effectively penalized shrinks towards $P \cap S_r(\tau) = E$. The structure

of this penalty is inseparable from the polyhedral-spherical representation: the polytope appears as the hard constraint domain, and the sphere appears as the soft constraint encoded in the penalty.

The reason the penalty recovers elements of E and not merely points of $P \cap S_r(\tau)$ in the set-theoretic sense is the vertex-located property established in §5.2: since $E_{nk}(G) = \text{vert } \Pi_{nk}(G)$, every point of $\Pi_{nk}(G) \cap S_r(\tau)$ is a vertex of $\Pi_{nk}(G)$ and hence an element of $E_{nk}(G)$.

The same holds for $B_2^n = \text{vert } [g_1, g_2]^n$. Thus $P \cap S_r(\tau) = E$, and the penalty (5.59) is a valid reformulation of the original discrete constraint: the global minimizer of $\Psi(\lambda, \cdot)$ over P , as $\lambda \rightarrow \infty$, is driven onto E .

Remark 5.3. The penalty term $\lambda(\|x - \tau\|^2 - r^2)^2$ is a biquadratic function of x and is not convex on $\mathbb{R}^n$ in general. Consequently, subproblem (5.60) is not a convex for general λ , and standard convex optimization guarantees do not apply. In particular, gradient descent on $\Psi(\lambda, \cdot)$ may converge to a local rather than global minimizer of (5.60). The method is therefore classified as approximate rather than exact. Its practical performance depends critically on the choice of the penalty schedule λ_j and the initialization strategy.

The behaviour of the method depends on how the parameter λ is chosen. In the trajectory regime, a non-decreasing sequence $0 \leq \lambda_0 < \lambda_1 < \dots \rightarrow \infty$ is prescribed.

At each stage j , subproblem (5.60) is solved with $\lambda = \lambda_j$, starting from the solution x^j obtained at the previous stage, with x^0 initialised from the polyhedral relaxation of §5.2. The iterates form a trajectory in P that, as λ_j increases, is progressively attracted towards $S_r(\tau)$ and ultimately converges to a point of E . This is analogous to a path-following method: the landscape of (5.60) deforms smoothly with λ , and the solution tracks the deforming minimizer. The total computational effort is distributed over the trajectory, and each individual subproblem is relatively well-conditioned because λ changes only incrementally.

In the large fixed penalty regime, a single large value λ is chosen from the out-set. When λ is very large, the penalty term dominates the objective and subproblem (5.60) effectively reduces to finding the point of P closest to the sphere $S_r(\tau)$. In this regime the original objective f plays a negligible role and the outcome depends almost entirely on the starting point. This regime is therefore not recommended as a standalone method; it may however serve as a final rounding step after the trajectory regime has brought the iterate close to E .

At every stage of the trajectory method, the relaxations of §§5.2–5.3 provide rigorous lower bounds on

$$z^* = \min_{x \in E} f(x).$$

Specifically,

$$z^* \geq z^D = \max \left\{ \min_{x \in P} f(x), \min_{x \in S_r(\tau)} f(x) \right\}, \quad (5.61)$$

where the first term argument of the maximum is the polyhedral lower bound from §5.2 and the second argument is the spherical lower bound from §5.3.

An upper bound is obtained by projecting the current iterate x^j onto E . As in §§5.4–5.5, this projection plays a dual role. On the one hand, it provides the upper bound: setting

On the one hand, this provides an upper bound

$$z_u = f(\hat{x}^j)$$

by finding the nearest vertex $\hat{x}^j$ of the set E to x^j via the projection operator

$$\hat{x}^j = Pr_E x^j.$$

On the other hand, the same candidate vertex is immediately subjected to the sufficient optimality check of Chapter 4: conditions (4.15)–(4.16) for $E_{nk}(G)$ or condition (4.32) for B_k^n .

Whenever this check succeeds, the algorithm terminates immediately with the closest vertex as a certified global solution, bypassing the remaining penalty stages.

The gap

$$(z_u - z^D) / |z^D|$$

serves as a practical stopping criterion: when it falls below a prescribed tolerance $\varepsilon > 0$, the iterate $Pr_E x^j$ is accepted as an approximate solution.

For $E_{nk}(G)$, the projection $Pr_E x^j$ is the nearest vertex $\hat{x}^j$ of $\Pi_{nk}(G)$ to x^j , found in $O(n \log n)$ by the sorting procedure of Lemma 4.3 (conditions (4.11)–(4.12)).

For B_k^n , the projection

$$\hat{x}^j = Pr_{B_k^n} x^j$$

is the coordinatewise rounding of Lemma 4.13 (condition (4.30)):

$$y_i^* = \begin{cases} g_1 & f \ x_i^j \leq (g_1 + g_k) / 2; \\ g_k & \text{otherwise.} \end{cases} \quad (5.62)$$

computed in $O(n)$ operations.

Algorithm 5.1. (*Penalty-Function Method*).

Input: configuration $E = P \cap S_r(\tau)$; objective f ; penalty sequence $\lambda_0 < \lambda_1 < \dots < \lambda_T$; tolerance $\varepsilon > 0$.

Step 0. Solve the polyhedral relaxation: $x^0 \leftarrow \arg \min_{x \in P} f(x)$.

Set lower bound $z^D \leftarrow f(x^0)$.

Compute $Pr_E x^0 \in E$, set upper bound $z_u \leftarrow f(Pr_E x^0)$ and check optimality conditions. Set $j \leftarrow 0$.

Step 1. Solve subproblem (5.60) with $\lambda = \lambda_j$, starting from x^j , to obtain x^{j+1} .

Step 2. Compute

$$\hat{x}^{j+1} = Pr_E x^{j+1};$$

by the projection procedure appropriate to the configuration class.

Update

$$z_u \leftarrow \min \{z_u, f(\hat{x}^{j+1})\}.$$

Check the sufficient optimality conditions for $\hat{x}^{j+1}$; if satisfied, return $\hat{x}^{j+1}$ as a certified global solution.

Update lower bound z^D from (5.61).

Step 3. If $(z_u - z^D) / |z^D| \leq \varepsilon$, return $\hat{x}^{j+1}$ as the approximate solution.

Otherwise, set $j \leftarrow j + 1$ and go to Step 1.

Output: approximate solution $\hat{x}^j \in E$ with gap

$$(z_u - z^D) / |z^D| \leq \varepsilon,$$

or exact solution certified by the optimality conditions of Chapter 4.

Theorem 5.5. (*Convergence of the trajectory*).

Let $f \in C^1(\mathbb{R}^n)$ and let $\{x^j\}$ be a sequence of global minimizers of (5.60) for $\lambda = \lambda_j \rightarrow \infty$. Then every limit point of $\{x^j\}$ belongs to $P \cap S_r(\tau) = E$.

Moreover, if $\{x^j\}$ converges to some $x^* \in E$, then x^* is a local minimizer of f on E .

Proof. Since P is compact, the sequence $\{x^j\}$ has at least one limit point $\bar{x}$. For any $\varepsilon > 0$, if

$$(\|\bar{x} - \tau\|^2 - r^2)^2 \geq \varepsilon^2$$

then $\Psi(\lambda_j, x^j) \rightarrow \infty$, contradicting the fact that

$$\Psi(\lambda_j, x^j) \leq \Psi(\lambda_j, y) = f(y)$$

for any fixed $y \in E$.

Hence $\bar{x} \in S_r(\tau)$. Since $\bar{x} \in P$ and $P \cap S_r(\tau) = E$, we conclude $\bar{x} \in E$. Local optimality of $\bar{x}$ follows by continuity of f and the fact that any neighbourhood of $\bar{x}$ in E is also a neighbourhood of the corresponding iterate for large j .

Remark 5.4. Theorem 5.5 assumes global minimizers of the non-convex subproblem (5.60). In practice, only local minimizers are available. The trajectory regime mitigates this by providing warm starts: when λ_k increases only slightly from λ_{j-1} , the landscape of (5.60) changes smoothly and the local minimizer tracked from stage to stage is less likely to be far from the global one. This is the principal practical advantage of the trajectory regime over the large fixed penalty regime.

For the multipermutation configuration $E_{nk}(G)$, the polytope is $P = \Pi_{nk}(G)$, the centre is $\tau = (\bar{\tau}, \dots, \bar{\tau})$, where $\bar{\tau} = S_1/n$, and $r = \sqrt{S_2 - S_1^2/n}$.

The penalty function (5.59) becomes

$$\Psi(\lambda, x) = f(x) + \lambda \sum_{i=1}^n ((x_i - \bar{\tau})^2 - r^2)^2. \quad (5.63)$$

The conditional gradient method is particularly effective for subproblem (5.60): its linearization step minimizes a linear function over $\Pi_{nk}(G)$, which by Lemma 4.1 (conditions (4.7)–(4.8)) is solved in $O(n \log n)$ by sorting. The derivative of (5.63) with respect to x is

$$\Psi'(x) = f'(x) + 4\lambda(\|x - \bar{\tau}\|^2 - r^2)(x - \bar{\tau}), \quad (5.64)$$

so each evaluation of $\Psi'(x)$ costs $O(n)$ beyond the cost of evaluating $f'(x)$.

The projection step is the nearest vertex of $\Pi_{nk}(G)$ to the current iterate, found in $O(n \log n)$ by conditions (4.11)–(4.12) of Lemma 4.3. The optimality check via conditions (4.15)–(4.16) requires only a comparison of orderings at cost $O(n \log n)$ and is performed at each projection step at negligible additional cost.

For the Boolean configuration $B_2^n = \{g_1, g_2\}^n$, the polytope is the hyperrectangle $Q_2^n = [g_1, g_2]^n$, the circumsphere has centre $\tau = (\bar{\tau}, \dots, \bar{\tau})$ where $\bar{\tau} = (g_1 + g_2) / 2$, and radius $r = \sqrt{n} (g_2 - g_1) / 2$.

The spherical penalty term simplifies as follows. On $[g_1, g_2]^n$ one has

$$\|x - \tau\|^2 - r^2 = \sum_{i=1}^n (x_i - g_1)(x_i - g_2).$$

This quantity equals zero if and only if each $x_i \in \{g_1, g_2\}$, that is, if and only if $x \in B_2^n$. The penalty function (5.59) thus becomes

$$\Psi(\lambda, x) = f(x) + \lambda \sum_{i=1}^n (x_i - g_1)(x_i - g_2). \quad (5.65)$$

The linearization step of the conditional gradient method over $[g_1, g_2]^n$ is trivial: the minimum of a linear function is attained by rounding each component independently to g_1 or g_2 by condition (4.29), computed in $O(n)$ operations.

The projection

$$\hat{x}^j = Pr_{B_2^n} x^j$$

is the coordinatewise rounding (5.62), computed in $O(n)$.

The derivative of (5.65) with respect to x is

$$\Psi'(x) = f'(x) + 2\lambda \sum_{i=1}^n (x_i - g^1)(x_i - g^2)(2x_i - g^1 - g^2),$$

where the subscript i denotes the i -th component, computed in $O(n)$ operations beyond the cost of evaluating $f'(x)$.

The optimality check (4.32) for $\hat{x}^j$ costs $O(n)$. No oracle for the permutohedron is required, making the Boolean specialization significantly more efficient than the multipermutation case.

Remark 5.5. In the linear case $f(x) = \langle c, x \rangle$, the Penalty-Function Method (PFM) of this section and the polyhedral-spherical cutting-plane method of §5.7 address the same class of problems but from opposite directions. The cutting-plane

method is exact: it iteratively tightens a polyhedral relaxation using spherical information to construct deep cuts. The PFM is approximate: it embeds the sphere in the objective via a penalty and solves a sequence of smooth subproblems. The two approaches are complementary. The PFM provides a high-quality initial upper bound and a warm start; the cutting-plane method can then certify optimality or improve the solution. For non-linear f , the PFM is the primary tool.

In summary, the penalty-function method converts the polyhedral-spherical representation directly into an algorithmic device: the polytope P constrains the search domain, and the sphere $S_r(\tau)$ penalizes deviations from the configuration E . The projection onto E plays the same dual role as in §§5.4–5.5: it provides both the upper bound and the input for the sufficient optimality check. For $E_{nk}(G)$ the projection costs $O(n \log n)$ and the optimality check costs $O(n \log n)$; for B_k^n both cost $O(n)$. The trajectory regime provides a principled approximate solution with rigorous lower and upper bounds at every stage, connecting the relaxation theory of §§5.2–5.3 with the combinatorial rounding tools of §§5.4–5.5.

The optimization methods developed in this chapter form a complete and coherent toolkit for Euclidean combinatorial optimization. Polyhedral and spherical relaxations provide lower bounds exploiting complementary geometric features of the carrier domain; branch-and-bound schemes integrate both relaxations into globally convergent algorithms whose subproblem structure is preserved under branching; and the penalty-function method converts the polyhedral-spherical representation directly into a smooth optimization device admitting rigorous two-sided bounds at every stage.

The efficiency of all methods derives from the same structural source established in Chapters 3 and 4: the vertex-located property ensures that the nearest-configuration projection reduces to sorting for multipermutation configurations and to coordinatewise comparison for Boolean ones. The geometric and analytic framework developed throughout the monograph thus translates directly into practically implementable algorithms, demonstrating that the Euclidean approach to combinatorial optimization yields both theoretical depth and computational transparency.

Conclusions

The present monograph develops a systematic mathematical theory at the interface of combinatorial optimization, convex analysis, and Euclidean geometry. Its central subject is the behaviour of functions defined on finite point configurations embedded in Euclidean space, and the exploitation of this behaviour for the solution of combinatorial optimization problems. The unifying principle throughout is the following: by choosing the Euclidean immersion so that the image of the combinatorial feasible set is vertex-located, one transforms a discrete optimization problem into a continuous one without losing any information, and opens the door to the full apparatus of convex analysis.

The geometric and analytical foundations of the theory rest on a precise study of finite point configurations in Euclidean space. Four interconnected frameworks were developed in Chapter 1: the metric, affine, and convex structure of configurations; decompositions induced by families of parallel hyperplanes and their connection to the facial geometry of the convex hull; decompositions by strictly convex level surfaces, leading to spherically located and polyhedral-spherical configurations; and functional representations through systems of inequalities. A fundamental result established in Chapter 1 is that every spherically located configuration is automatically vertex-located. This result links smooth and polyhedral convex geometry and underlies the treatment of hyperspherical extensions in Chapter 2 and the polyhedral-spherical methods of Chapter 5. The functional representation framework developed in Chapter 1 provides the analytic carriers on which convex extensions are constructed and supplies the starting point for continuous approaches to combinatorial optimization.

The theory of convex extensions of functions defined on finite vertex-located configurations constitutes the theoretical core of the monograph. Three principal results were established in Chapter 2. First, existence theorems were proved for convex, strongly convex, and differentiable extensions of an arbitrary function from a vertex-located configuration to its convex hull and to larger super-carrier domains. The decisive structural property is vertex location itself: since no configuration point is a convex combination of the others, convexity imposes

no internal compatibility constraints on the prescribed functional values, and extensions exist for any data whatsoever. Second, constructive schemes were developed for each extension class: the minimal convex extension via extremal interpolation over vertex decompositions of the carrier polyhedron, strongly convex extensions via quadratic regularization, and differentiable extensions via infimal convolution with smooth kernels. Third, a coherent extremal framework was established, yielding a hierarchy of three lower bound estimates of increasing sharpness: a subgradient bound requiring only convexity, a gradient bound requiring differentiability, and a quadratic bound requiring strong convexity. Sufficient optimality conditions complementing each bound were formulated in terms of the normal cone structure of the carrier polyhedron. Together, the bounds and optimality conditions constitute a complete duality between lower estimation and optimality verification, making the framework self-contained and directly applicable to combinatorial optimization.

Five principal classes of combinatorial configurations were introduced and studied in Chapter 3, formalized as finite point sets in Euclidean space through the Euclidean immersion framework: permutation configurations, multipermutation configurations, partial multipermutation configurations, partial permutation configurations, and unbounded partial multipermutation configurations including Boolean sets. For each class, the vertex-located property was established, the associated combinatorial polytope was characterized in detail through dimension formulae, vertex criteria, face lattice structure, and adjacency relations, and polyhedral-spherical representations were identified where they hold. A unifying picture emerged: vertex-locatedness is unconditional for permutation, multipermutation, and Boolean configurations; it holds stratum-wise for partial multipermutation configurations; and it holds for partial permutation configurations if and only if the sequence length equals the ground set cardinality plus one. This classification determines precisely the scope of applicability of the convex extension theory of Chapter 2 to each configuration class and provides the combinatorial and geometric foundation for the analytical constructions of Chapters 4 and 5.

The abstract existence theory of Chapter 2 was made explicit and constructive for each of the principal configuration classes in Chapter 4. Convex extension schemes were developed for functions defined on multipermutation, partial permutation, and Boolean configurations, yielding analytic formulas for the extensions in terms of the combinatorial parameters of the configuration. The extremal properties of linear, separable, distance-type, and general convex functions were investigated systematically. For linear and separable functions on full and partial configurations, minimization was reduced to ordering and rearrangement operations governed by explicit exchange principles. For quadratic functions on partial permutation configurations, the extremal problem was shown to reduce to a classical assignment problem. For unbounded and Boolean configurations, coordinate independence yields closed-form solutions of complete transparency.

Lower bounds and sufficient optimality conditions were derived across the entire hierarchy, revealing how convex continuation interacts with discrete admissibility constraints at each level. The results of Chapter 4 transform the existence guarantees of Chapter 2 into concrete, computable tools: for each specific configuration class and function type, the practitioner obtains not merely the guarantee that an extension exists, but its explicit form and the resulting explicit bounds on the minimum.

Three classes of optimization methods were developed in Chapter 5 for Euclidean combinatorial optimization problems, all grounded in the geometric structure established in the preceding chapters. Polyhedral relaxation, which replaces the combinatorial configuration by its convex hull, and spherical relaxation, which replaces it by the circumscribed hypersphere, were developed and compared. A theoretical result establishes that neither relaxation dominates the other in general, and the combined lower bound, taken as their maximum, is strictly tighter than either component. A branch-and-bound scheme was developed for multipermutation and Boolean configurations, integrating both relaxations into a complete global optimization algorithm whose subproblems preserve their polyhedral-spherical structure under branching. A penalty-function method for optimization on polyhedral-spherical configurations was developed, providing rigorous two-sided estimates at every stage. Throughout Chapter 5, the efficiency of all methods derives from the same structural source: the vertex-located property ensures that the nearest-vertex computation reduces to sorting for multipermutation configurations and to coordinatewise comparison for Boolean ones.

Across all five chapters, a single conceptual thread connects the geometric, analytic, and algorithmic results. The Euclidean immersion of a combinatorial feasible set produces a finite point configuration whose vertex-located property is both the condition for the existence of convex extensions and the source of efficiency for all optimization methods. Vertex location ensures that the discrete problem carries no convex redundancy; convex extension transfers the discrete data into a continuous carrier without loss; and the geometry of the carrier polytope governs both the sharpness of the extremal estimates and the computational cost of the methods. This tight coupling between combinatorial structure and continuous analysis is the distinctive feature of the Euclidean approach to combinatorial optimization, and it is this coupling that the present monograph develops in full generality and applies systematically.

Several directions of natural continuation suggest themselves. The configuration classes studied in this monograph constitute the most important and well-structured members of the broader family of Euclidean combinatorial sets. Extending the convex extension theory and the associated extremal analysis to further classes, including cyclic permutations, signed permutations, permutation matrices, and composite configurations generated by iterative inclusion, remains an open and natural direction. For each new class, the key steps are the same:

establish vertex location, characterize the carrier polytope, and construct explicit extension schemes for relevant function classes.

The configuration classes studied in this monograph are defined by occupancy and repetition constraints on coordinate positions. A natural structural generalization considers configurations with additional algebraic constraints, such as even and odd permutations, permutations with forbidden patterns, or configurations invariant under prescribed symmetry groups. For such classes the convex hull geometry is governed not only by occupancy rules but also by algebraic admissibility conditions, and characterizing their vertex structure, face lattices, and polyhedral-spherical representations requires new geometric arguments. The convex extension framework developed here provides the analytic template for these investigations, and the extremal ordering principles of Chapter 4 suggest that analogous rearrangement inequalities may hold in the algebraically constrained setting.

The extremal theory of Chapter 4 was developed for scalar-valued objective functions. Its extension to vector-valued objectives and multicriteria combinatorial optimization is both natural and practically important: the geometric structure of the carrier polytope can be exploited in the multicriteria setting, and the convex extension framework may provide new tools for the analysis of efficient frontiers and Pareto optimality conditions on combinatorial domains.

The optimization methods of Chapter 5 were developed for objective functions admitting convex extensions. When the objective is non-convex, the penalty construction of Chapter 2 and generalized convex extension theory provide a route to equivalent convex reformulations. Developing computational implementations of these methods and studying their practical performance on benchmark problems from scheduling, assignment, and logistics is a direction of both theoretical and applied interest. The polyhedral-spherical structure of combinatorial configurations further suggests connections to gradient methods on manifolds and to semidefinite relaxations, which may yield new relaxation hierarchies and approximation guarantees for broader problem classes.

The theoretical framework developed in this monograph has direct relevance to a broad range of applied domains, including scheduling and sequencing, assignment and matching, logistics and routing, and combinatorial data analysis. In each of these settings, the feasible set of a discrete optimization problem can often be modelled as a combinatorial point configuration of one of the classes studied here, making the convex extension and relaxation methods of Chapters 4 and 5 immediately applicable. Systematic computational studies on benchmark problem families from these domains, combined with the development of efficient implementations of the branch-and-bound and penalty-function algorithms, represent a natural and practically important direction for future work. Such studies would both validate the theoretical framework and reveal the structural features of applied problems that determine which relaxation strategy performs best in practice.

The results established in this monograph contribute to a broader mathematical programme whose significance extends beyond the specific configuration classes studied here. The central insight, that vertex location of a finite point set is both necessary and sufficient for the unrestricted existence of convex extensions, places a purely combinatorial condition at the foundation of a continuous analytic theory. This insight reframes the classical tension between discrete and continuous mathematics: rather than approximating discrete structures by continuous ones, the Euclidean immersion framework embeds them exactly, preserving all combinatorial information while gaining access to the full power of convex analysis.

The practical significance of the framework is equally substantial. Combinatorial optimization problems of permutation and assignment type are among the most frequently encountered in operations research and applied mathematics. Scheduling problems, in which tasks must be ordered or assigned to resources; routing problems, in which sequences of visits must be optimized; and data analysis problems, in which observations must be ranked or clustered according to combinatorial criteria, all reduce naturally to optimization over configuration classes of the type studied here. The convex extension and relaxation methods developed in this monograph provide, for the first time, a unified theoretical basis for addressing this entire family of problems through continuous convex analytic tools, without sacrificing the exactness of the combinatorial formulation. This unification suggests that the Euclidean approach to combinatorial optimization may serve as a theoretical foundation for a new generation of algorithms that are simultaneously rigorous in their mathematical guarantees and efficient in their computational implementation.

The mathematical theory developed in this monograph is, in a deeper sense, a theory of how discrete structure generates continuous geometry. Every combinatorial feasible set carries within itself, through its Euclidean immersion, a convex geometric object whose properties are entirely determined by the combinatorial rules that define the set. Reading this geometry correctly, constructing its analytic extensions, and exploiting its structure algorithmically are the three acts of which the present work is composed. The authors hope that the framework developed here will serve not only as a foundation for further theoretical work on Euclidean combinatorial optimization, but also as an invitation to explore the rich and largely uncharted territory that lies at the boundary between discrete mathematics, convex geometry, and continuous optimization.

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Abstract

The present monograph develops a systematic mathematical theory at the interface of combinatorial optimization, convex analysis, and Euclidean geometry. Its central subject is the behaviour of functions defined on finite point configurations embedded in Euclidean space, and the exploitation of this behaviour for the solution of combinatorial optimization problems.

By choosing the Euclidean immersion so that the image of the combinatorial feasible set coincides with the vertex set of its convex hull, a discrete optimization problem is transformed into a continuous one without any loss of information. This structural property is both the condition for the existence of convex extensions and the ultimate source of efficiency of all optimization methods developed in the monograph.

The theoretical core of the work is a complete theory of convex extensions of functions defined on vertex-located configurations: existence theorems for convex, strongly convex, and differentiable extensions; constructive schemes yielding explicit analytic formulas; and a hierarchy of computable extremal estimates and optimality conditions. This theory is applied to the principal classes of finite point configurations arising as Euclidean images of combinatorial sets, including permutation, multipermutation, partial permutation, and Boolean configurations, whose associated polytopes are characterized in full. Building on this foundation, the monograph develops optimization methods including polyhedral and spherical relaxations, branch-and-bound schemes, and penalty function methods, each grounded in the specific geometry of the combinatorial feasible set rather than in general-purpose algorithmic principles.

The book is addressed to researchers and graduate students in combinatorial optimization, convex analysis, and discrete mathematics, as well as to practitioners working on optimization problems in scheduling, assignment, logistics, and engineering design.

Keywords: *combinatorial optimization, finite point configuration, combinatorial point configuration, combinatorial polytope, vertex-located set, polyhedral-spherical configuration, convex extension, extremal estimate, nonlinear optimization.*

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Building on this foundation, the monograph develops optimization methods including polyhedral and spherical relaxations, branch-and-bound schemes, and penalty function methods, all exploiting the geometry of combinatorial feasible sets.

The book is intended for researchers and graduate students in combinatorial optimization, convex analysis, and discrete mathematics, as well as practitioners working on optimization problems in scheduling, assignment, logistics, and engineering design.



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